Monday Lecture 1
Welfare Economics

August 6, 2012

An exchange economy

- There are ℓ **commodities** indexed $h = 1, ..., \ell$. The **commodity space** is $X = \mathbf{R}^{\ell}$.
- There are m **agents** (consumers), indexed $i = 1, ..., m$, with **consumption set** X, a **utility function** $U_i : X \to \mathbf{R}$ and initial **endowment** of commodities $\mathbf{e}_i \in X$.
- The m-tuple $\mathcal{E} = \{(U_i, \mathbf{e}_i)\}_{i=1}^m$ is called an **exchange economy**.
- An **allocation** is an array $\mathbf{x} = (\mathbf{x}_1, ..., \mathbf{x}_m) \in X^m$.
- An allocation $\mathbf{x} \in X^m$ is **attainable** if

$$\sum_{i=1}^{m} \mathbf{x}_i = \sum_{i=1}^{m} \mathbf{e}_i .$$

Walrasian equilibrium

- A **Walrasian equilibrium** consists of an attainable allocation $\mathbf{x}^*$ and a price vector $\mathbf{p}^* \neq 0$ such that, for every $i = 1, \ldots, m$, $\mathbf{x}_i^*$ maximizes U_i on the budget set

$$B_i(\mathbf{p}^*) = \{\mathbf{x}_i \in X_i : \mathbf{p}^* \cdot \mathbf{x}_i \leq 0\}.$$

- We call $\mathbf{x}^*$ a **Walras allocation** if $(\mathbf{x}^*, \mathbf{p}^*)$ is a Walrasian equilibrium, for some price vector $\mathbf{p}^* \neq 0$.
- An attainable allocation $\mathbf{x}$ is said to be **weakly Pareto efficient** if there does not exist an attainable allocation $\mathbf{y}$ such that $U_i(\mathbf{y}_i) > U_i(\mathbf{x}_i)$, for every $i = 1, \ldots, m$.
- An attainable allocation x is said to be (**strongly**) **Pareto efficient** if there does not exist an attainable allocation $\mathbf{y}$ such that $U_i(\mathbf{y}_i) \geq U_i(\mathbf{x}_i)$, for every $i = 1, \ldots, m$, and $U_i(\mathbf{y}_i) > U_i(\mathbf{x}_i)$ for at least one i.

First Theorem of Welfare Economics

Theorem (First Theorem of Welfare Economics)

A Walras allocation is weakly Pareto efficient.

Proof.

Let $(\mathbf{x}, \mathbf{p})$ be an equilibrium and suppose, contrary to what we want to prove, that $\hat{\mathbf{x}}$ is attainable and is strictly preferred to $\mathbf{x}$ by every agent i. Then $\mathbf{p} \cdot \hat{\mathbf{x}}_i > \mathbf{p} \cdot \mathbf{e}_i$ for every i, so $\sum_i \mathbf{p} \cdot \hat{\mathbf{x}}_i > \sum_i \mathbf{p} \cdot \mathbf{e}_i$, contradicting $\sum_i \hat{\mathbf{x}}_i = \sum_i \mathbf{e}_i$. □

Example

As an example of an equilibrium that is weakly but not strongly Pareto efficient, use the Edgeworth Box with "thick" indifference curves.

First Theorem continued

- Agent i is **locally non-satiable** if, for any point $\mathbf{x}_i$ in the consumption set X_i and any $\varepsilon > 0$ there is a consumption bundle $\mathbf{x}_i' \in X_i$ such that $\|\mathbf{x}_i' - \mathbf{x}_i\| < \varepsilon$, $\mathbf{x}_i' \gg \mathbf{x}_i$, and $U_i(\mathbf{x}_i') > U_i(\mathbf{x}_i)$.

Theorem (First Theorem of Welfare Economics)

A Walras allocation $\mathbf{x}$ *is strongly Pareto efficient if every agent is locally non-satiable.*

Proof.

Let $(\mathbf{x}, \mathbf{p})$ be an equilibrium and suppose, contrary to what we want to prove, that $\mathbf{x}$ is attainable and is weakly preferred to $\mathbf{x}$ by every agent i and strictly preferred by some agent i. Local non-satiability implies that $\mathbf{p} \cdot \mathbf{x}_i \geq \mathbf{p} \cdot \mathbf{e}_i$ for every i and the inequality is strict for some i, so $\sum_i \mathbf{p} \cdot \mathbf{x}_i > \sum_i \mathbf{p} \cdot \mathbf{e}_i$, contradicting $\sum_i \mathbf{x}_i = \sum_i \mathbf{e}_i$. □

Second Theorem of Welfare Economics

- An attainable allocation $\mathbf{x}$ can be **decentralized** if there exists a price vector $\mathbf{p} \neq 0$ such that, for every $i = 1, ..., m$,

$$U_i(\mathbf{y}_i) > U_i(\mathbf{x}_i) \Longrightarrow \mathbf{p} \cdot \mathbf{y}_i > \mathbf{p} \cdot \mathbf{x}_i.$$

- Let $P_i(\mathbf{x}_i)$ denote the set of points that is preferred to $\mathbf{x}_i$ by agent i, that is,

$$P_i(\mathbf{x}_i) = \{\mathbf{y}_i \in X : U_i(\mathbf{y}_i) > U_i(\mathbf{x}_i)\},$$

for every $i = 1, ..., m$.

Theorem (Second Theorem of Welfare Economics)

Suppose that $\mathbf{x}^*$ is a weakly efficient allocation and suppose that U_i is continuous and l.n.s. and $P_i(\mathbf{x})$ is convex for every $i = 1, ..., m$. Then $\mathbf{x}^*$ can be decentralized using a price vector $\mathbf{p}^* \neq 0$.

Proof of Second Theorem

- Let $Z_i = P_i(\mathbf{x}_i^*) - \{\mathbf{x}_i^*\}$ for each i and let $Z = \sum_i Z_i$. Then

  (a) Z is nonempty because U_i is l.n.s.,
  (b) Z_i is open because U_i is continuous, and
  (c) Z_i is convex because $P_i(\mathbf{x}_i^*)$ is convex.

- We claim that $0 \notin Z$ because $\mathbf{x}^*$ is weakly Pareto efficient. If not, there exist vectors $\mathbf{z}_i \in Z_i$ such that $\sum_i \mathbf{z}_i = 0$. Then $\mathbf{y} = \mathbf{x}^* + \mathbf{z}$ is an attainable allocation and $U_i(\mathbf{y}_i) > U_i(\mathbf{x}_i^*)$ for every i, a contradiction.

Theorem (Minkowski lemma)

Let S be a nonempty, open and convex set and suppose $0 \notin S$. Then there exists a vector $\mathbf{p} \in \mathbf{R}^\ell$ such that $\mathbf{p} \neq 0$ and $\mathbf{p} \cdot \mathbf{x} > 0$ for any $\mathbf{x} \in S$.

Proof of Second Theorem continued

- By the Minkowski lemma, there exists a vector $\mathbf{p}^* \neq 0$ such that $\mathbf{p}^* \cdot z > 0$ for any $z \in Z$.
- The continuity of U_i implies that $0 \in \overline{Z}_i$ for all i.
- This inequality implies that $\mathbf{p}^* \cdot z_i > 0$ for any $z_i \in Z_i$, as required.
- This shows that $\mathbf{x}^*$ is decentralized using the price vector $\mathbf{p}^*$.

An economy with risk

- Assume a finite number of states of nature, $s = 1, ..., S$, with common probability distribution $\pi(s)$.
- Agent i has a vNM utility function $U_i : X \to \mathbf{R}$.
- The **commodity space** is $X^S = \{\mathbf{x} : S \to X\}$, where $\mathbf{x}(s)$ denotes the bundle of contingent commodities delivered in state s. The **endowment** of agent i is $\mathbf{e}_i \in X^S$.
- An exchange economy is defined by the m-tuple $\mathcal{E} = \{(U_i, \mathbf{e}_i)\}_{i=1}^m$.
- An **allocation** for the economy is now an m-tuple $\mathbf{x} = \{\mathbf{x}_i\}_{i=1}^m$ such that $\mathbf{x}_i \in X^S$ for each i. The allocation $\mathbf{x}$ is **attainable** if $\sum_{i=1}^m \mathbf{x}_i = \sum_{i=1}^m \mathbf{e}_i$.
- A **Walrasian equilibrium** consists of an attainable allocation $\mathbf{x}^*$ and a price vector $\mathbf{p}^* \in X^S$ such that, for every $i = 1, ..., m$, the consumption bundle $\mathbf{x}_i^*$ maximizes $\sum_{s=1}^S \pi(s) U_i(\mathbf{x}_i(s))$ in the budget set

$$B_i(\mathbf{p}^*) = \left\{ \mathbf{y}_i \in X^S : \mathbf{p}^* \cdot \mathbf{y}_i \leq \mathbf{p}^* \cdot \mathbf{e}_i \right\}.$$

Arrow securities

- Securities are traded before the true state of nature is revealed.
- The ℓ goods are traded on spot markets after the true state is revealed.
- An **Arrow security** promises one unit of account in some state s and nothing in other states.
- We assume that there is a complete set of Arrow securities, one for each state s.
- Let z_{is} denote agent i's demand (positive or negative) for Arrow security s and let

$$\mathbf{z}_i = (z_{i1}, ..., z_{is}, ..., z_{iS})$$

denote the vector of security demands for agent i. The set of possible security demands is denoted by $Z \equiv \mathbf{R}^S$.

Allocations

- A **security allocation** is an m-tuple $\mathbf{z} = (\mathbf{z}_1, ..., \mathbf{z}_i, ..., \mathbf{z}_m) \in Z^m$. An allocation of contingent commodities is an m-tuple $\mathbf{x} = (\mathbf{x}_1, ..., \mathbf{x}_i, ..., \mathbf{x}_m) \in \left(X^S\right)^m$, where $\mathbf{x}_i$ is the vector of demands for contingent commodities for agent i.
- An **allocation** consists of an order pair $(\mathbf{x}, \mathbf{z}) \in \left(X^S\right)^m \times Z^m$, where $\mathbf{x}$ is an allocation of contingent commodities and $\mathbf{z}$ is the allocation of securities.
- An allocation $(\mathbf{x}, \mathbf{z})$ is **attainable** if

$$\sum_{i=1}^{m} (\mathbf{x}_i, \mathbf{z}_i) = \sum_{i=1}^{m} (\mathbf{e}_i, 0) .$$

- Let let $\mathbf{q} = (q_1, ..., q_s, ..., q_S)$ q_s denote the vector of Arrow security prices, let $\mathbf{p}(s) \neq 0$ denote the vector of commodity prices in state s and let $\mathbf{p} = (\mathbf{p}(1), ..., \mathbf{p}(s), ...\mathbf{p}(S))$.

Equilibrium with Arrow securities

Definition

An **equilibrium with Arrow securities** consists of an attainable allocation $(\mathbf{x}^*, \mathbf{z}^*)$ and a **price system** $(\mathbf{p}^*, \mathbf{q}^*)$ such that, for each agent i, the ordered pair $(\mathbf{x}_i^*, \mathbf{z}_i^*)$ maximizes

$$\sum_{s=1}^{S} \pi(s) U_i(\mathbf{x}_i(s))$$

subject to the budget constraints

$$\mathbf{q}^* \cdot \mathbf{z}_i \leq 0$$

and

$$\mathbf{p}^*(s) \cdot \mathbf{x}_i(s) \leq \mathbf{p}^*(s) \cdot \mathbf{e}_i(s) + z_{is}, \ \forall s.$$

Equivalence

Theorem

If $(\mathbf{x}^*, \mathbf{p}^*)$ *is a Walrasian equilibrium, then* $(\mathbf{x}^*, \mathbf{z}^*, \mathbf{p}^*, \mathbf{q}^*)$ *is an equilibrium with Arrow securities, where* $\mathbf{q}^* = (1, ..., 1)$ *and* $\mathbf{z}_i^*$ *is defined by*

$$z_{is}^* = \mathbf{p}^*(s) \cdot (\mathbf{x}_i^*(s) - \mathbf{e}_i(s)), \ \forall s.$$

Conversely, if $(\mathbf{x}^*, \mathbf{z}^*, \mathbf{p}^*, \mathbf{q}^*)$ *is an equilibrium with Arrow securities, then* $(\mathbf{x}^*, \mathbf{p})$ *is an equilibrium with complete markets, where* $\mathbf{p}$ *is defined by*

$$\mathbf{p}(s) = q_s \mathbf{p}^*(s), \ \forall s.$$

Proof of Arrow's Theorem

Suppose that $(\mathbf{x}^*, \mathbf{p}^*)$ is a Walrasian equilibrium and let $\mathbf{z}^*$ and $\mathbf{q}^*$ be defined by

$$q_s = 1$$

and

$$z_{is}^* = \mathbf{p}^*(s) \cdot (\mathbf{x}_i^*(s) - \mathbf{e}_i(s)),$$

for any $s = 1, ...S$ and $i = 1, ..., m$. We want to show that $(\mathbf{x}^*, \mathbf{z}^*, \mathbf{p}^*, \mathbf{q}^*)$ is an equilibrium with Arrow securities.

Step 1: Note that, for every state s,

$$\sum_{i=1}^{m} z_{is}^* = \sum_{i=1}^{m} \mathbf{p}^*(s) \cdot (\mathbf{x}_i^*(s) - \mathbf{e}_i(s)) = 0$$

because $\mathbf{x}^*$ is attainable, so $\sum_{i=1}^{m} \mathbf{z}_i^* = 0$ and, hence, $(\mathbf{x}^*, \mathbf{z}^*)$ is attainable.

Proof continued

Step 2: For every agent i,

$$\mathbf{q}^* \cdot \mathbf{z}_i^* = \sum_{s=1}^{S} \mathbf{p}^*(s) \cdot (\mathbf{x}_i^*(s) - \mathbf{e}_i(s)) \leq 0,$$

since $\mathbf{x}_i^* \in B_i(\mathbf{p}^*)$. Also, for every i,

$$\mathbf{p}^*(s) \cdot \mathbf{x}_i^*(s) \leq \mathbf{p}^*(s) \cdot \mathbf{e}_i(s) + z_{is}^*, \ \forall s.$$

Hence, $(\mathbf{x}_i^*, \mathbf{z}_i^*)$ belongs to the budget set of agent i.
Step 3: Now suppose that $(\mathbf{x}_i, \mathbf{z}_i)$ belongs to the budget set of agent i. The budget constraints imply that

$$\sum_{s=1}^{S} \mathbf{p}^*(s) \cdot (\mathbf{x}_i(s) - \mathbf{e}_i(s)) \leq \sum_{s=1}^{S} z_{is} \leq 0,$$

so $\mathbf{x}_i$ belongs to the Walrasian budget set $B_i(\mathbf{p}^*)$. Since $\mathbf{x}_i^*$ maximizes expected utility over the budget set $B_i(\mathbf{p}^*)$, $(\mathbf{x}_i^*, \mathbf{z}_i^*)$ must maximize expected utility over the budget set in the equilibrium with Arrow securities.

Proof continued

Step 4: Suppose that $(\mathbf{x}^*, \mathbf{z}^*, \mathbf{p}^*, \mathbf{q}^*)$ is an equilibrium with Arrow securities. Define a price vector $\mathbf{p}$ for a Walrasian equilibrium by putting

$$\mathbf{p}(s) = q_s^* \mathbf{p}^*(s), \ \forall s.$$

Clearly, $\mathbf{x}^*$ is attainable because $(\mathbf{x}^*, \mathbf{z}^*)$ is attainable. Also, $\mathbf{x}_i^*$ belongs to the budget set $B_i(\mathbf{p})$ because

$$\begin{aligned}
\mathbf{p} \cdot (\mathbf{x}_i^* - \mathbf{e}_i) &= \sum_{s=1}^{S} \mathbf{p}(s) \cdot (\mathbf{x}_i^*(s) - \mathbf{e}_i(s)) \\
&= \sum_{s=1}^{S} q_s^* \mathbf{p}^*(s) \cdot (\mathbf{x}_i^*(s) - \mathbf{e}_i(s)) \\
&\leq \sum_{s=1}^{S} q_s^* z_{is}^* \leq 0
\end{aligned}$$

from the budget constraints of the equilibrium with Arrow securities.

Proof continued

Step 5: Now suppose that $\mathbf{x}_i$ belongs to $B_i(\mathbf{p})$. Define $\mathbf{z}_i$ by putting

$$z_{is} = \mathbf{p}^*(s) \cdot (\mathbf{x}_i(s) - \mathbf{e}_i(s)), \ \forall s.$$

Then

$$\begin{aligned} \mathbf{q}^* \cdot \mathbf{z}_i &= \sum_{s=1}^{S} q_s^* \mathbf{p}^*(s) \cdot (\mathbf{x}_i^*(s) - \mathbf{e}_i(s)) \\ &= \sum_{s=1}^{S} \mathbf{p}(s) \cdot (\mathbf{x}_i^*(s) - \mathbf{e}_i(s)) \\ &= \mathbf{p} \cdot (\mathbf{x}_i^* - \mathbf{e}_i) \leq 0, \end{aligned}$$

so $(\mathbf{x}_i, \mathbf{z}_i)$ belongs to the budget set of the equilibrium with Arrow securities. Since $(\mathbf{x}_i^*, \mathbf{z}_i^*)$ maximizes expected utility in the budget of the equilibrium with Arrow securities, agent i must prefer $\mathbf{x}_i^*$ to $\mathbf{x}_i$. Thus, $\mathbf{x}_i^*$ is optimal in the budget set $B_i(\mathbf{p})$. This completes the proof that $(\mathbf{x}^*, \mathbf{p})$ is a Walrasian equilibrium.

Necessary conditions for optimal risk sharing

- If $U_i(x_{is})$ is C^1, a necessary first-order condition for optimality is that

$$\pi_s U_i'(x_{is}^*) = \lambda_i p_s^*$$

  for every state $s = 1, ..., S$.

- Eliminating λ_i we get

$$\frac{\pi_s U_i'(x_{is}^*)}{\pi_{s'} U_i'(x_{is'}^*)} = \frac{\lambda_i p_s^*}{\lambda_i p_{s'}^*} = \frac{p_s^*}{p_{s'}^*}$$

  for any states s, s'.

- This immediately implies that, for any pair of agents i and j,

$$\frac{\pi_s U_i'(x_{is}^*)}{\pi_{s'} U_i'(x_{is'}^*)} = \frac{\pi_s U_j'\left(x_{js}^*\right)}{\pi_{s'} U_j'\left(x_{js'}^*\right)}.$$

The Borch conditions

- Canceling probabilities on both sides, we have the **Borch conditions:**

$$\frac{U_i'\left(x_{is}^*\right)}{U_i'\left(x_{is'}^*\right)} = \frac{U_j'\left(x_{js}^*\right)}{U_j'\left(x_{js'}^*\right)}.$$

- The Borch conditions are the necessary and sufficient conditions for efficient risk sharing when the utility functions $u_i\left(\mathbf{x}_i\right)$ are concave and continuously differentiable.
- They imply the necessity of **coinsurance**, that is, every agent's consumptions moves in the same direction between states.
- If utility functions are strictly concave, coinsurance has a striking implication: efficient risk sharing implies that each agent's consumption is a function of the total endowment.

Demand functions

Let $\mathcal{E} = \{(X_i, \mathbf{e}_i, U_i)\}$ be an exchange economy satisfying the following properties:

- $X_i = \mathbf{R}^\ell_+$ and $\mathbf{e}_i \gg 0$ for any $i = 1, ..., m$.
- $U_i : \mathbf{R}^\ell_+ \to \mathbf{R}$ is increasing, continuous and strictly quasi-concave.
- Let $P = \left\{\mathbf{p} \in \mathbf{R}^\ell_+ : p_h \gg 0, p_\ell = 1\right\}$ and let $\bar{P} =$cl$P$ denote the closure of P. For any $\mathbf{p} \in P$, let $B_i(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}_i) = \{\mathbf{x}_i \in X_i : \mathbf{p} \cdot \mathbf{x}_i \leq \mathbf{p} \cdot \mathbf{e}_i\}$ and let

$$\xi_i(\mathbf{p}) = \arg\max\left\{U_i(\mathbf{x}_i) : \mathbf{x}_i \in B_i(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}_i)\right\}.$$

- $\xi_i(\mathbf{p})$ is a singleton for any $\mathbf{p} \in P$ and $\xi_i : P \to \mathbf{R}^\ell_+$ is a well-defined function. Moreover, ξ_i is continuous on P and, for any sequence $\{\mathbf{p}^q\}$ in P converging to $\mathbf{p}^0 \in \bar{P} \backslash P$, $\|\xi_i(\mathbf{p}^q)\| \to \infty$.

Excess demand functions

- Define the **individual excess demand function** $z_i : P \to \mathbf{R}^\ell$ for agent i by putting

$$z_i(\mathbf{p}) = \xi_i(\mathbf{p}) - \mathbf{e}_i,$$

  for every $\mathbf{p} \in P$ and define the **aggregate excess demand function** for $\mathcal{E}$, denoted by $\mathbf{z} : P \to \mathbf{R}^\ell$, by putting

$$z(\mathbf{p}) = \sum_{i=1}^{m} z_i(\mathbf{p}),$$

  for every $\mathbf{p} \in P$.

- Under the maintained assumptions, $\mathbf{z} : P \to \mathbf{R}^\ell$ is well defined for the pure exchange economy $\mathcal{E}$. The function $\mathbf{z}$ is continuous and satisfies the boundary condition

$$\|\mathbf{z}(\mathbf{p}^q)\| \to \infty$$

  for any sequence $\{\mathbf{p}^q\}$ in P such that $\mathbf{p}^q \to \mathbf{p}^0 \in \bar{P} \setminus P$.

Regular economies

Let $U \subset \mathbf{R}^n$ be an open set, let $f : U \rightarrow \mathbf{R}^n$ be a function, and suppose that $\mathbf{x} \in U$ is a solution of the equation

$$f(\mathbf{x}) = 0.$$

Then we say that $\mathbf{x}$ is **locally unique** (or a locally unique solution) if there is some open set $V \subset U$ such that $\mathbf{x} \in V$ and there does not exist $\mathbf{y} \neq \mathbf{x}$ in V such that $f(\mathbf{y}) = 0$. The following theorem is often used to establish local uniqueness.

Theorem

Inverse Function Theorem. *Let $U \subset \mathbf{R}^n$ be open and $f : U \rightarrow \mathbf{R}^n$ be C^r, $1 \leq r \leq \infty$, at $\mathbf{x}$. If the matrix of derivatives $\nabla f(\mathbf{x})$ is nonsingular (invertible), then there is an open set $V \subset \mathbf{R}^n$ such that $f(\mathbf{x}) \in V$ and a C^r function $f^{-1} : V \rightarrow \mathbf{R}^n$ such that $f^{-1}(f(\mathbf{y})) = \mathbf{y}$ on a neighborhood of $\mathbf{x}$. Moreover,*

$$\nabla f^{-1}(f(\mathbf{x})) = [\nabla f(\mathbf{x})]^{-1}.$$

A C^1 inverse at $f(\mathbf{x})$ can exist only if $\nabla f(\mathbf{x})$ is nonsingular.

Regularity

- We assume that $z : P \longrightarrow \mathbf{R}^\ell$ is a member of the class C^r for $1 \leq r \leq \infty$, normalize the price vector by putting $p_\ell = 1$ and denote the vector of excess demands of the first $\ell - 1$ goods by

$$\hat{z}(\mathbf{p}) = (z_1(\mathbf{p}), ..., z_{\ell-1}(\mathbf{p}))\,.$$

- The excess demand function z satisfies Walras' law, that is,

$$\mathbf{p} \cdot z(\mathbf{p}) = 0,$$

  for any $\mathbf{p} \in P$. If all but one market clears, the remaining market must clear also.

- A normalized price vector $\mathbf{p} = (p_1, ..., p_{\ell-1}, 1)$ constitutes a **Walrasian equilibrium** if and only if it solves the system of $\ell - 1$ equations in $\ell - 1$ unknowns

$$\hat{z}(\mathbf{p}) = 0.$$

Local uniqueness

Definition

An equilibrium price vector $\mathbf{p} = (p_1, ..., p_{\ell-1}, 1)$ is **regular** if the $(\ell - 1) \times (\ell - 1)$ matrix of price effects $\nabla \hat{z}(\mathbf{p})$ is non-singular, that is, has rank $\ell - 1$. If every normalized equilibrium price vector is regular we say that the economy is **regular**.

Theorem

Any regular (normalized) equilibrium price vector $\mathbf{p} = (p_1, ..., p_{\ell-1}, 1)$ *is* **locally unique**. *That is, for some* $\varepsilon > 0$ *and any* $\mathbf{p}' \neq \mathbf{p}$ *such that* $p'_\ell = p_\ell = 1$ *and* $\|\mathbf{p}' - \mathbf{p}\| < \varepsilon$, $z(\mathbf{p}') \neq 0$. *Moreover, if the economy is regular, then the number of normalized equilibrium price vectors is finite.*

Genericity I

A property is said to be **generic** if it holds for all parameters outside a negligible set, for example, a set of measure zero. We want to show that regularity is such a generic property.

Theorem (Transversality)

Suppose that $f : \mathbf{R}^m \times \mathbf{R}^p \longrightarrow \mathbf{R}^n$ *If the* $n \times (m + p)$ *matrix* $\nabla f(\mathbf{x}; \mathbf{q})$ *has rank* n *whenever* $f(\mathbf{x}, \mathbf{q}) = 0$ *then for almost every* $\mathbf{q}$, *the* $n \times m$ *matrix* $\nabla_{\mathbf{x}} f(\mathbf{x}; \mathbf{q})$ *has rank* n *whenever* $f(\mathbf{x}; \mathbf{q}) = 0$.

Write the excess demand function $\hat{z}(\mathbf{p}; \mathbf{e})$ to show the dependence on the vector of endowments $\mathbf{e} = (\mathbf{e}_1, \ldots, \mathbf{e}_m)$.

Theorem (Rank Condition)

For any $\mathbf{p}$ *and* $\mathbf{e}$, *the rank of* $\nabla_{\mathbf{e}} \hat{z}(\mathbf{p}; \mathbf{e})$ *is* $\ell - 1$.

Genericity II

The next proposition follows directly from the Transversality Theorem and the Rank Condition.

Theorem

For almost every vector of initial endowments $\mathbf{e} = (\mathbf{e}_1, ..., \mathbf{e}_m)$, *the economy defined by* $\{(u_i, \mathbf{e}_i)\}_{i=1}^{m}$ *is regular.*

Asset economies

- There two dates $t = 0, 1$ and a finite number of states $s = 0, 1, ..., S$. The state of nature is unknown at date 0 but has a common probability distribution $\pi(s)$; the true state is revealed at date 1.
- There are $\ell + 1$ goods, indexed by $h = 0, ..., \ell$ so the commodity space is $\mathbf{R}^{(\ell+1)(S+1)}$.
- There are $m + 1$ economic agents, indexed by $i = 0, ..., m$, characterized by the consumption set $\mathbf{R}_{+}^{(\ell+1)(S+1)}$, an endowment $\mathbf{e}_i \in \mathbf{R}_{+}^{(\ell+1)(S+1)}$ and a utility function $U_i : \mathbf{R}_{+}^{(\ell+1)(S+1)} \to \mathbf{R}$.
- There is a finite set of assets, indexed by $k = 0, 1, ..., K$, all in zero net supply.
- Asset k is defined by a payoff vector $\mathbf{a}_k = (a_{k0}, ..., a_{kS})$, where a_{ks} is the return in terms of the numeraire good $h = 0$ in state s.
- The payoff matrix $\mathbf{A} = [a_{ks}]_{(K+1)\times(S+1)}$ characterizes the possibilities of trade between periods and states.

Allocations and prices

- An **allocation** for the economy is an array $(\mathbf{x}, \mathbf{z}) = \{(\mathbf{x}_i, \mathbf{z}_i)\}_{i=1}^{m}$ such that $\mathbf{x}_i \in \mathbf{R}_+^{(\ell+1)(S+1)}$ and $\mathbf{z}_i \in \mathbf{R}^K$ for each $i = 1, ..., m$.
- An allocation $(\mathbf{x}, \mathbf{z})$ is **attainable** if

$$\sum_{i=1}^{m} \mathbf{x}_i = \sum_{i=1}^{m} \mathbf{e}_i \text{ and } \sum_{i=1}^{m} \mathbf{z}_i = \mathbf{0}.$$

- A **price system** consists of a pair of price vectors $(\mathbf{p}, \mathbf{q})$, where $\mathbf{p} \in \mathbf{R}^{(\ell+1)(S+1)}$ and $\mathbf{q} \in \mathbf{R}^K$.
- Assume free disposal, so that $(\mathbf{p}, \mathbf{q}) \geq (\mathbf{0}, \mathbf{0})$, w.l.o.g.
- Partition the consumption bundle $\mathbf{x}_i$ for agent i into the sub-bundles $\mathbf{x}_i(s)$ at states $s = 0, 1, ..., S$, where $s = 0$ denotes the first date, and write $\mathbf{x}_i = (\mathbf{x}_i(0), \mathbf{x}_i(1), ..., \mathbf{x}_i(S))$.
- Similarly, partition the price system $\mathbf{p}$ into the sub-vectors $\mathbf{p}(s)$, for $s = 0, 1, ..., S$, and write $\mathbf{p} = (\mathbf{p}(0), ..., \mathbf{p}(S))$.

Equilibrium

Definition

An **equilibrium** for the economy consists of an attainable allocation $(\mathbf{x}^*, \mathbf{z}^*)$ and a price system $(\mathbf{p}, \mathbf{q})$ such that, for every agent $i = 1, ..., m$,

$(\mathbf{x}_i^*, \mathbf{z}_i^*)$ *maximizes* $U_i(\mathbf{x}_i)$ *subject to the constraints*

$$\mathbf{q} \cdot \mathbf{z}_i \leq 0$$

$$\mathbf{p}(s) \cdot \mathbf{x}_i(s) \leq \mathbf{p}(s) \cdot \mathbf{e}_i(s) + \sum_{k=1}^{K} z_k a_{ks},$$

for $s = 0, 1, ..., S$.

Assumptions

(A.1) U_i is continuous and quasi-concave on $\mathbf{R}_{+}^{(\ell+1)(S+1)}$ and the range of U_i can be extended to $\mathbf{R} \cup \{-\infty\}$.

(A.2) $\mathbf{e}_i \gg \mathbf{0}$.

(A.3) U_i is increasing in the numeraire good at every state $s = 1, ..., S$ at date 1.

(A.4) Free assets give rise to arbitrage: there exists a portfolio $\hat{\mathbf{z}} \in \mathbf{R}^{K+1}$ such that $\hat{\mathbf{z}}\mathbf{A} > \mathbf{0}$.

(D.1) $\mathbf{A}$ has full row rank.

(D.2) U_i is C^2, $DU_i \gg \mathbf{0}$ and $D^2 U_i$ is negative definite on $\mathbf{R}_{++}^{(\ell+1)(S+1)}$.

(D.3) The closure of the indifference curves of U_i do not intersect the boundary of $\mathbf{R}_{+}^{(\ell+1)(S+1)}$.

(S) The asset market is incomplete: $K < S$.

(CS) Every set of $K + 1$ columns of $\mathbf{A}$ are linearly independent and there exists a portfolio $\hat{\mathbf{z}}$ such that $\mathbf{a}(s) \cdot \hat{\mathbf{z}} \neq 0$, for all states $s = 1, ..., S$.

Spot market equilibrium relative to z^

Theorem

An equilibrium exists if (A.1) through (A.4) are satisfied.

Definition

Let $\hat{\mathbf{z}}$ be a fixed but arbitrary profile of assets satisfying

$$\sum_{i=1}^{m} \hat{\mathbf{z}}_i = \mathbf{0}$$

and define a **spot market equilibrium relative to $\hat{\mathbf{z}}$** to be an attainable allocation $(\hat{\mathbf{x}}, \hat{\mathbf{z}})$ and a price system $(\hat{\mathbf{p}}, \hat{\mathbf{q}})$ such that, for every agent $i = 1, \dots, m$, $\hat{\mathbf{x}}_i$ maximizes $U_i(\mathbf{x}_i)$ subject to the constraints

$$\hat{\mathbf{q}} \cdot \mathbf{z}_i \leq 0$$

$$\hat{\mathbf{p}}(s) \cdot \mathbf{x}_i(s) \leq \hat{\mathbf{p}}(s) \cdot \mathbf{e}_i(s) + \sum_{k=1}^{K} z_{ik} a_{ks},$$

The space of economies

- Let $\mathcal{E} \subset \mathbf{R}_{++}^{(S+1)(\ell+1)(m+1)}$ be an open set of endowments of each of the m agents and assume that $\mathcal{E}$ is bounded and that the closure of E does not intersect the boundary of $\mathbf{R}_{++}^{(S+1)(\ell+1)(m+1)}$.
- $\mathcal{U}$ is assumed to be a finite dimensional manifold of utility functions satisfying the assumptions previously assumed and sufficiently rich in perturbations so that, if $U_i \in \mathcal{U}$, then $U_i + \varepsilon f \in \mathcal{U}$ for $\varepsilon > 0$ sufficiently small, where f is any smooth function.
- The space of economies is identified with the parameters in $\mathcal{E} \times \mathcal{U}^m$.
- A set of economies $D \subset \mathcal{E} \times \mathcal{U}^m$ is said to be **generic** if it is an open dense subset of $\mathcal{E} \times \mathcal{U}^m$ with a null complement. (A null set is here interpreted to be a set of measure zero).

Regularity

Theorem

If (A1) through (A.4) and (D.1) through (D.3) are satisfied, then for any choice of utilities $\mathbf{U} \in \mathcal{U}$*, there is a generic set* $E(\mathbf{U})$ *of endowments in* $\mathcal{E}$ *such that for every economy* $(\mathbf{e}, \mathbf{U})$ *with* $e \in E(\mathbf{U})$*, the set of competitive equilibria is a continuously differentiable function of the endowment allocation* $\mathbf{e}$*.*

Theorem

If (A1) through (A.4) and (D.1) through (D.3) are satisfied, then there is a generic set of economies $D \subset \mathcal{E} \times \mathcal{U}^m$ *on which*
(i) the set of competitive equilibria is finite and is a continuously differentiable function of the endowment and utility assignment $(\mathbf{e}, \mathbf{U})$*;*
(ii) the spot market competitive equilibrium corresponding to any competitive portfolio allocation is, locally, a continuously differentiable function of the portfolio allocation $\mathbf{z}$*.*

Constrained inefficiency

- An attainable allocation $(\mathbf{x}, \mathbf{z})$ is **Pareto efficient** if there does not exist an attainable allocation $(\mathbf{x}', \mathbf{z}')$ such that $U_i(\mathbf{x}_i') \geq U_i(\mathbf{x}_i)$ for $i = 1, ..., m$ and the inequality is strict for some i.

Proposition

- *If the asset market is incomplete (S) and if (A.1) through (A.4) and (D.1) through (D.3) are satisfied, then for any economy* $(\mathbf{e}, \mathbf{U}) \in D$, *a generic set, all competitive equilibria are Pareto inefficient.*

- An equilibrium allocation $(\mathbf{x}^*, z^*)$ is said to be **constrained efficient** if there does not exist a feasible portfolio profile $\hat{\mathbf{z}}$ and spot market equilibrium relative to $\hat{\mathbf{z}}$, say $(\hat{\mathbf{x}}, \hat{\mathbf{z}}, \hat{\mathbf{p}}, \hat{\mathbf{q}})$ such that $U_i(\hat{\mathbf{x}}_i) \geq U_i(\mathbf{x}_i^*)$ for $i = 1, ..., m$ and the inequality is strict for some i.

A generic result

Theorem

Suppose that $0 < 2\ell \leq m < S\ell$. If the asset market is incomplete (S) and if (A.1) through (A.4), (D.1) through (D.3) and (CS) are satisfied, then for any economy $(\mathbf{e}, \mathbf{U}) \in D$, a generic set all competitive equilibria are constrained inefficient as long as there are at least two assets, $K + 1 \geq 2$. If $K + 1 \geq 3$, this remains true even if the reallocation of assets must satisfy the asset budget constraint for each individual at the equilibrium asset prices.