Monday Lecture 2
Optimal Intermediation

August 6, 2012

Intermediation and crises

- Bryant (1980) and Diamond and Dybvig (1983) made four contributions:
  (i) A **maturity structure** of bank assets, in which less liquid assets earn higher returns
  (ii) A theory of **liquidity preference**, modeled as uncertainty about the timing of consumption
  (iii) The representation of a bank as an intermediary that provides **insurance** to depositors against liquidity (preference) shocks
  (iv) An explanation of bank runs by depositors
- In the case of Diamond and Dybvig (1983), the bank runs are modeled as the result of **self-fulfilling prophecies or panics**; in the case of Bryant (1980), they are modeled as the result of **fundamentals**

The basic setup

- There are three dates, $t = 0, 1, 2$. At each date there is an all-purpose consumption/investment good
- There are two types of assets:
  - the liquid asset (also called the *short asset*) is a constant returns to scale technology that takes one unit of the good at date t and converts it into one unit of the good at date $t + 1$, where $t = 0, 1$;
  - the illiquid asset (also called the *long asset*) is a constant returns to scale technology that takes one unit of the good at date 0 and transforms it into $R > 1$ units of the good at date 2; if the long asset is liquidated prematurely at date 1 then it pays $0 < r < 1$ units of the good for each unit invested
- There is a large number of ex ante identical economic agents. Each consumer has an endowment of one unit of the good at date 0 and nothing at the later dates

Consumers

- With probability λ an agent is an **early consumer**, who only values consumption at date 1; with probability $(1-\lambda)$ he is a **late consumer** who only values consumption at date 2. The agent's (random) utility function $u(c_1, c_2)$ is defined by

$$u(c_1, c_2) = \begin{cases} U(c_1) & \text{w.pr. } \lambda \\ U(c_2) & \text{w.pr. } 1-\lambda, \end{cases}$$

  where $c_t \geq 0$ denotes consumption at date $t = 1, 2$ and $U(\cdot)$ is a neoclassical utility function (increasing, strictly concave, twice continuously differentiable)
- "Law of large numbers": λ is the fraction of early consumers; $1-\lambda$ is the fraction of late consumers
- Uncertainty about the agent's type gives rise to **liquidity preference**

Market equilibrium

- We assume that there exists a market on which an agent can sell his holding of the long asset at date 1 after he discovers his true type
- At date 0, a consumer invests in a portfolio (x, y) subject to the budget constraint

$$x + y \leq 1$$

- At date 1 he discovers his type. An early consumer liquidates his portfolio and consume the proceeds:

$$c_1 = y + Px$$

  where P is the price of the long asset
- A late consumer rebalance his portfolio (w.l.o.g., we assume he holds only the long asset):

$$c_2 = \left(x + \frac{y}{P}\right) R$$

- At date 0, the consumer chooses (x, y) to maximize expected utility

$$\lambda U\left(y + Px\right) + \left(1 - \lambda\right) U\left[\left(x + \frac{y}{P}\right) R\right]$$

The value of the market

- In equilibrium, the price of the long asset must be $P = 1$, so the consumer's consumption is

$$c_1 = x + Py = x + y = 1$$

at date 1 and

$$c_2 = \left(x + \frac{y}{P}\right) R = (x + y)R = R$$

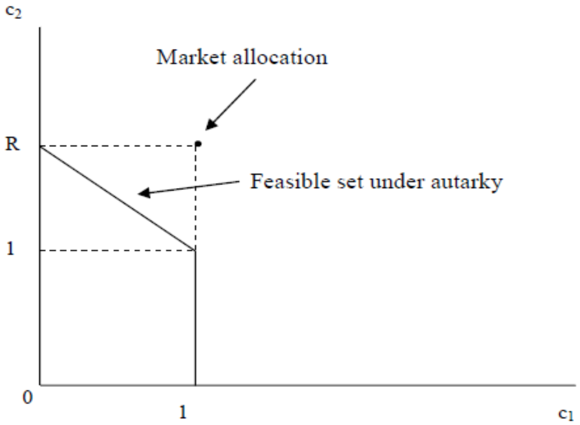
at date 2

- Then the equilibrium expected utility is

$$\lambda U(1) + (1 - \lambda) U(R)$$

- In autarky, consumption is $(y, y + R(1 - y))$ for $0 \leq y \leq 1$.The market allocation $(c_1, c_2) = (1, R)$ dominates every feasible autarkic allocation

Figure 1: Market allocation and feasible set under autarky



The efficient solution

- The market solution is inefficient because it does not allow trades contingent on the agent's type, i.e., insurance against liquidity shocks. We represent the efficient allocation as the solution to a planner's problem
- Feasibility:

$$x + y = 1 \tag{1}$$

$$\lambda c_1 \leq y \tag{2}$$

$$\lambda c_1 + (1 - \lambda) c_2 \leq Rx + y \tag{3}$$

- The planner's objective is to choose the investment portfolio (x, y) and the consumption allocation (c_1, c_2) to maximize the typical investor's expected utility

$$\lambda U(c_1) + (1 - \lambda) U(c_2),$$

subject to the various feasibility conditions (1) – (3)

Solving the planner's problem

- W.l.o.g., we can assume the short asset is used to provide consumption at date 1 and the long asset is used to provide consumption at date 2:

$$c_1 = \frac{y}{\lambda};$$
$$c_2 = \frac{Rx}{1-\lambda}$$

- Substituting these expressions for consumption into the objective function the planner's problem is

$$\max_{0 \le y \le 1} \left\{ \lambda U\left(\frac{y}{\lambda}\right) + (1-\lambda)\, U\left(\frac{R(1-y)}{1-\lambda}\right) \right\} \tag{4}$$

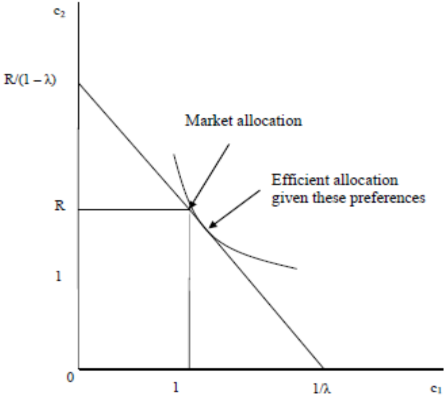
- A necessary condition for an interior optimum is

$$U'(c_1) = RU'(c_2)\, R \tag{5}$$

The inefficiency of the market solution

- The feasible allocations for the planner's problem are defined by the equation
$$(c_1, c_2) = \left(\frac{y}{\lambda}, \frac{R(1-y)}{(1-\lambda)} \right)$$
- The market allocation corresponds to putting $y = \lambda$:
$$(c_1, c_2) = \left(\frac{y}{\lambda}, \frac{R(1-y)}{(1-\lambda)} \right) = (1, R)$$
- The first-order condition for optimality (5) at this point is
$$U'(1) = U'(R)R$$
This will be satisfied in the special case of log utility function, but not in general
- Suppose that
$$U(c) = \frac{1}{1-\sigma} c^{1-\sigma}$$
Then $\sigma > 1$ implies that (c_1, c_2) satisfies $c_1 > 1$ and $c_2 < R$ and, conversely, $\sigma < 1$ implies that $c_1 < 1$ and $c_2 > R$

Figure 2: Inefficiency of the market solution



Complete markets

- A consumer can purchase date-t goods at a price q_t at date 0, for delivery conditional on being type $t = 1, 2$. The consumer's budget constraint is

$$q_1 \lambda c_1 + q_2 p \left(1 - \lambda\right) c_2 \leq 1, \tag{6}$$

  where $p = \frac{P}{R}$ is price of date-2 goods in terms of date-1 goods
- The individual chooses $\left(c_1, c_2\right)$ to maximize $\lambda U\left(c_1\right) + \left(1 - \lambda\right) U\left(c_2\right)$ subject to (6) and the solution must satisfy the first-order conditions

$$\lambda U'\left(c_1\right) = \mu q_1 \lambda$$

$$\left(1 - \lambda\right) U'\left(c_2\right) = \mu q_2 p \left(1 - \lambda\right).$$

  Then

$$\frac{U'\left(C_1\right)}{U'\left(C_2\right)} = \frac{q_1}{q_2 p}$$

No-arbitrage conditions

- There are zero profits from investing in the short asset if and only if

$$q_1 = 1$$

- Similarly, there are zero profits from investing in the long asset if and only if

$$pq_2 = \frac{1}{R}$$

- These no-arbitrage conditions imply that

$$\frac{U'(C_1)}{U'(C_2)} = R,$$

the condition required for efficient risk sharing

The banking solution

- A bank takes a deposit of one unit of the good, invests it in a portfolio (x, y) and offers the consumer a **deposit contract** (c_1, c_2)
- Free entry into the banking sector and competition force banks to maximize the ex ante expected utility of the typical depositor subject to a zero-profit constraint
- The bank chooses the investment portfolio (x, y) and the consumption allocation (c_1, c_2) to maximize

$$\lambda U(c_1) + (1 - \lambda) U(c_2),$$

  subject to the feasibility conditions (1) – (3)
- The incentive-compatibility constraint is automatically satisfied because $c_1 \leq c_2$

Financial fragility (multiple equilibria)

- **Liquidation technology:** premature liquidation of the long asset yields $r \leq 1$ units of the good
- If all depositors withdraw at date 1, the liquidated value of bank assets is

$$rx + y \leq x + y = 1$$

- If $c_1 > rx + y$, the bank is insolvent and will be able to pay only a fraction of the promised amount c_1 and nothing will be left at date 2

	Run	No Run
Run	$(rx + y, rx + y)$	(c_1, c_2)
No Run	$(0, rx + y)$	(c_2, c_2)

- It is clear that if

$$0 < rx + y < c_1 < c_2$$

then (Run, Run) is an equilibrium and (No Run, No Run) is also an equilibrium

- Suspension of convertibility and the sequential service constraint

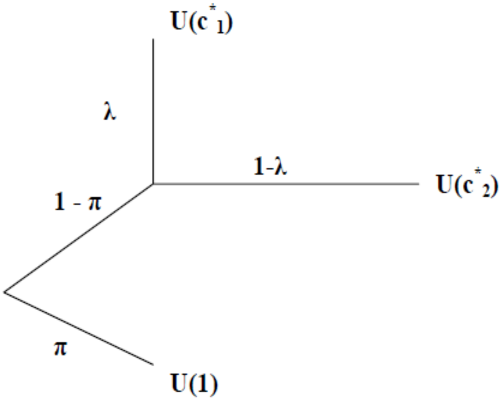
Equilibrium bank runs

The Diamond-Dybvig argument shows the possibility of an *unexpected* run

- A run cannot be predicted with certainty
- So, the best we can hope for is a bank run that occurs *with positive probability*
- A **sunspot** (random variable) takes two values, high and low, with probabilities π and $1-\pi$, respectively. Depositors run on the bank when the sunspot is "high" and not when it is "low"
- The bank chooses a portfolio (x, y) and a deposit contract (c_1, c_2), in the expectation that (c_1, c_2) is achieved only if the bank is solvent. In the event of a bank run, the typical depositor will receive the value of the liquidated portfolio $rx + y$ at date 1.

$$(\tilde{C}_1, \tilde{C}_2) = \begin{cases} (rx+y, rx+y) & \text{if } S = \text{"high"} \\ (c_1, c_2) & \text{if } S = \text{"low"} \end{cases}$$

Figure 3: Runs with positive probability



The optimal portfolio

- The expected utility of the representative depositor can be written

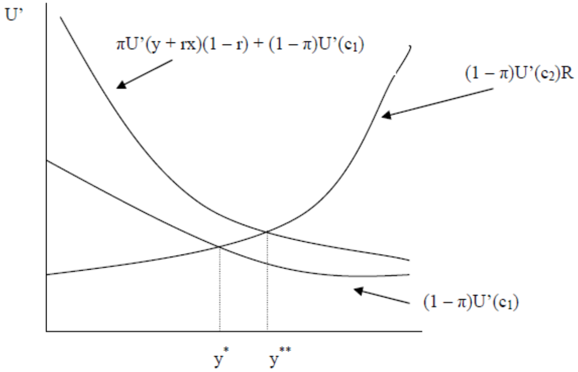
$$\pi U(y + rx) + (1 - \pi) \{\lambda U(c_1) + (1 - \lambda) U(c_2)\}.$$

- The optimal portfolio must satisfy the first-order condition

$$\pi U'(y + rx)(1 - r) + (1 - \pi) U'(c_1) = (1 - \pi) U'(c_2) R.$$

- If $\pi = 0$ then this reduces to the familiar condition $U'(c_1) = U'(c_2) R$.
- The possibility of a run ($\pi > 0$) increases the value of a marginal increase in y and hence increases the amount of the short asset held in the portfolio.

Figure 4: The optimal portfolio when runs are possible



The optimal deposit contract

- The bank chooses the deposit contract (c_1^*, c_2^*) to satisfy the first-order condition
$$U'(c_1^*) = RU'(c_2^*). \tag{7}$$
- Suppose for simplicity that $r = 1$ and relative risk aversion is greater than one. These conditions imply the possibility of a run
- The long asset now dominates the short asset so, without essential loss of generality, we can assume $y = 0$ and $x = 1$
- The deposit contract must solve the decision problem

$$\begin{array}{ll} \max & \lambda U(c_1) + (1-\lambda)U(c_2) \\ \text{s.t.} & R\lambda c_1 + (1-\lambda)\, c_2 \leq R \end{array}$$

Equilibrium without runs

- The bank can prevent a run by choosing a sufficiently "safe" contract:

$$c_1 \leq 1$$

- If we solve the problem

$$\begin{array}{ll} \max & \lambda U(c_1) + (1-\lambda) U(c_2) \\ \text{s.t.} & R(\lambda c_1) + (1-\lambda) c_1 \leq R \\ & c_1 \leq 1 \end{array}$$

we find the solution $(c_1^{**}, c_2^{**}) = (1, R)$

A characterization of regimes with and without runs

- If the bank anticipates a sunspot with probability π, it will be better to avoid runs if

$$\pi U(1) + (1-\pi)\left\{\lambda U(c_1^*) + (1-\lambda)\, U\left(c_2^*\right)\right\} > \lambda U(1) + (1-\lambda)\, U(R)$$

- The expected utility from the safe strategy $\lambda U(1) + (1-\lambda)\, U(R)$ lies between two values:

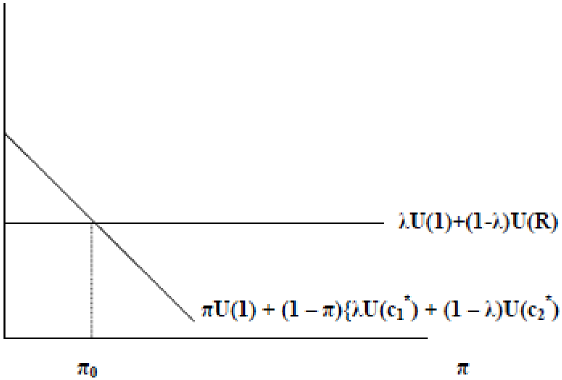
$$\begin{aligned} U\left(1\right) &< \lambda U(1) + (1-\lambda)\, U(R) \\ &< \lambda U(c_1^*) + (1-\lambda)\, U\left(c_2^*\right) \end{aligned}$$

- There exists a unique value $0 < \pi_0 < 1$ such that

$$\pi_0 U(1) + (1-\pi_0)\left\{\lambda U(c_1^*) + (1-\lambda)\, U\left(c_2^*\right)\right\} = \lambda U(1) + (1-\lambda)\, U(R$$

- The bank will prefer runs if and only if $\pi < \pi_0$

Figure 5: Equilibrium with positive probability of runs



Essential bank runs

- The *long asset* has a random return $\tilde{R}$ at date 2 given by

$$\tilde{R} = \begin{cases} R_H & \text{w. pr. } \pi_H \\ R_L & \text{w. pr. } \pi_L \end{cases}$$

  If the long asset is prematurely liquidated, it yields r units of the good at date 1. We assume that

$$R_H > R_L > r > 0$$

- Without loss of generality we put $c_2 = \infty$ and characterize the deposit contract by $c_1 = d$
- We consider only **essential** bank runs, that is, runs that cannot be avoided
- At date 1, the budget constraint requires $\lambda d \leq y$. If there is no run, late consumers receive

$$c_{2s} = \frac{R_s(1-y) + y - \lambda d}{1-\lambda}$$

Bankruptcy

- The *incentive constraint* requires

$$d \leq R_s(1-y)+y$$

- The necessary and sufficient condition for an *essential* bank run is that the incentive constraint is violated, that is,

$$d > R_s(1-y)+y$$

- There are three cases to consider:
  - the incentive constraint is never binding and bankruptcy never occurs;
  - bankruptcy is a possibility but the bank finds it optimal to choose a deposit contract and portfolio so that the incentive constraint is (just) satisfied;
  - the costs of distorting the choice of deposit contract and portfolio are so great that the bank finds it optimal to allow bankruptcy in the low state

Case I: The incentive constraint is not binding in equilibrium

- We solve the intermediary's decision problem without the incentive constraint and then check whether the constraint is binding or not
- The intermediary chooses y and d to maximize expected utility, assuming that there is no bank run
  - With probability λ, the depositor is an early consumer and receives d regardless of the state
  - With probability $1-\lambda$, the depositor is a late consumer and his consumption in state s is $R_s(1-y)+y-\lambda d$ divided by the number of late consumers $1-\lambda$
- Thus, the expected utility

$$\lambda U(d) + (1-\lambda)\left\{\pi_H U\left(\frac{R_H(1-y)+y-\lambda d}{1-\lambda}\right) + \pi_L U\left(\frac{R_L(1-y)+y-\lambda d}{1-\lambda}\right)\right\}$$

is maximized subject to $0 \leq y \leq 1$ and $\lambda d \leq y$

The optimal portfolio

- Assuming an interior solution, i.e., $0 < y < 1$, the optimal (y, d) is characterized by the necessary and sufficient first-order conditions:

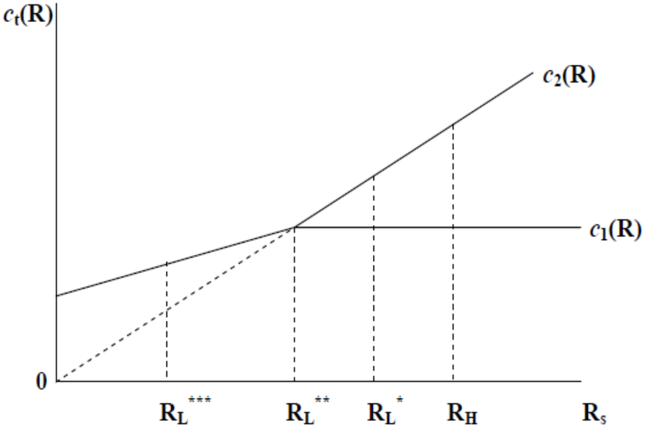
$$U'(d) - \left\{ \pi_H U' \left(\frac{R_H(1-y) + y - \lambda d}{1-\lambda} \right) + \pi_L U' \left(\frac{R_L(1-y) + y - \lambda d}{1-\lambda} \right) \right\} \geq 0,$$

and

$$\pi_H U' \left(\frac{R_H(1-y) + y - \lambda d}{1-\lambda} \right) (1 - R_H) + \pi_L U' \left(\frac{R_L(1-y) + y - \lambda d}{1-\lambda} \right) (1 - R_L) \leq 0,$$

with equality in each case if $\lambda d < y$

Figure 6: Consumption as a function of R



Equilibrium with a non-binding IC constraint

- A solution to these inequalities, (y^*, d^*), represents an equilibrium if

$$d^* \leq R_L(1 - y^*) + y^*$$

- Let U^* denote the maximized value of expected utility corresponding to (y^*, d^*)
- If the low state return R_L is sufficiently high, say $R_L = R_L^*$, then the incentive constraint is never binding
- Early consumers receive

$$c_{1s} = d = \frac{y}{\lambda}$$

and late consumers receive

$$c_{2s} = \frac{R_s(1 - y)}{(1 - \lambda)}$$

in each state $s = H, L$

Case II: The incentive constraint is binding in equilibrium

- Suppose that (y^*, d^*) does not satisfy the incentive constraint
- If the intermediary chooses not to default, the decision problem is to choose (y, d) to maximize

$$\lambda U(d) + (1 - \lambda)\left\{\pi_H U\left(c_H\right) + \pi_L U\left(c_L\right)\right\}$$

subject to the feasibility constraints

$$0 \leq y \leq 1 \text{ and } \lambda d \leq y$$

and the incentive constraints

$$c_{2s} = \frac{R_s\left(1 - y\right) + y - \lambda d}{1 - \lambda} \geq d, \text{ for } s = H, L$$

- The incentive constraint will only bind in the low state $s = L$

- Substituting for $c_{2L} = d$, the expression for expected utility can be written as

$$\lambda U(d) + (1-\lambda)\left\{\pi_H U\left(\frac{R_H(1-y)+y-\lambda d}{1-\lambda}\right) + \pi_L U(d)\right\},$$

  where the incentive constraint implies that $d \equiv R_L(1-y)+y$
- In this case, the first-order condition that characterizes the choice of y takes the form

$$\lambda U'(d)(1-R_L) + (1-\lambda)\left\{\pi_H U'\left(\frac{R_H(1-y)+y-\lambda d}{1-\lambda}\right) \times \left(\frac{1-R_H-1+\lambda R_L}{1-\lambda}\right) + \pi_L U'(d)(1-R_L)\right\} \leq 0,$$

  with equality if $\lambda d < y$
- Let (y^{**}, d^{**}) denote the solution to this problem and let U^{**} denote the corresponding maximized expected utility

Case III: The incentive constraint is violated in equilibrium

- Default in the low state implies expected utility is

$$\pi_H \left\{ \lambda U(d) + (1-\lambda) U\left(\frac{R_H(1-y) + y - \lambda d}{1-\lambda} \right) \right\} + \pi_L U\left(r(1-y) + y\right)$$

- The FOCs for an optimum take the form

$$\pi_H \left\{ \lambda U'(d) - \lambda U'\left(\frac{R_H(1-y) + y - \lambda d}{1-\lambda} \right) \right\} \geq 0,$$

$$\pi_H U'\left(\frac{R_H(1-y) + y - \lambda d}{1-\lambda} \right)(1-R_H) + \pi_L U'\left(r(1-y) + y\right)(1-R_L) \leq 0,$$

with equality if $\lambda d < y$

- Let (d^{***}, y^{***}) and U^{***} denote the solution and maximum value

- If $R_L = R_L^{***}$ then bankruptcy occurs in the low state and both early and late consumers receive the same consumption in the low state:

$$c_{1L} = c_{2L} = y + R_L \left(1 - y\right) < d$$

- In the high state,

$$c_{1H} = d \text{ and } c_{2H} = \frac{R_H \left(1 - y\right)}{1 - \lambda}$$

- This is an equilibrium solution only if

$$d^{***} > R_L(1 - y) + y,$$

and

$$U^{***} > U^{**}$$

- The first condition guarantees that the incentive constraint is violated
- The second condition guarantees that default is preferred to solvency