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WHY ARE ASSET RETURNS MORE VOLATILE DURING RECESSIONS? A THEORETICAL EXPLANATION

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Why Are Asset Returns More Volatile during Recessions? A Theoretical Explanation

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Abstract

During recession, many macroeconomic variables display higher levels of volatility. We show how introducing an AR(1)-ARCH(1) driving process into the canonical Lucas consumption CAPM framework can account for the empirically observed greater volatilty of asset returns during recessions. In particular, agents' joint forecasting of levels and time-varying second moments transforms symmetric-volatility driving processes into asymmetric-volatility endogenous variables. Moreover, numerical examples show that the model can indeed account for the degree of cyclical variation in both bond and equity returns in the U.S. data. Finally, we argue that the underlying mechanism is not specific to financial markets, and has the potential to explain cyclical variation in the volatilities of a wide variety of macroeconomic variables.

1 Introduction

When it rains, it pours. In times of recession, not only are many macroeconomic variables in bad shape with respect to their levels, but they are also plagued by higher levels of volatility. But what is it about recessions that seems to exacerbate volatility?

This paper is based upon Chapter 1 of my Ph.D. thesis at the Universitat Pompeu Fabra. Many thanks to Marianne Baxter, Jordi Caballe, Antonio Ciccone, Jim Costain, Jean-Pierre Danthine, Harris Dellas, Bernard Dumas, Albert Marcet, Antonio Novales and Rafael Repullo for valuable comments and suggestions. I also benefitted greatly from comments of seminar participants at the Bank of England, Carlos III, UPF and Warwick Business School, as well as the CEPR/LIFE/Wharton Conference on International Finance and Economic Activity (Vougliameni, 2000), the World Congress of the Econometric Society (Seattle, 2000) and the European Finance Association Annual Meeting (London, 2000). The usual caveat applies.

In order to address the question of why volatility might be higher during recessions, a framework must allow volatility to vary over time. That is, not only the first but also the second moments of relevant processes must change over time. In such a framework, agents need to base their optimal decisions upon forecasts both of future levels and of future variances. The main contribution of this paper is to show that it is precisely agents' optimal joint forecasting of levels and variances which leads to volatility which is asymmetric over the business cycle. In more technical terms, symmetric-volatility forcing processes are transformed into asymmetric-volatility endogenous variables. High-volatility recessions and low-volatility expansions may emerge endogenously due to agents' optimal behavior when forecasting levels and variances jointly

What is it about forecasting variances and levels jointly that produces asymmetries? In the ARCH-forecasting framework developed in this paper, agents use observations on the innovations u_t to forecast the level of some process, while using u_t^2 to forecast its variance. This means that each realization of the shock carries two messages: one on the level and another on the variance. These two pieces of news may reinforce one another, but they can also contradict one another. For example, a large negative shock will hold two pieces of bad news: bad news on the level due to $u_t < 0$ and bad news on the variance due to u_t^2 large. Thus, any dismay about the bad news of a large negative shock will be amplified: *when it rains, it pours*. A large and positive shock, on the other hand, will carry both good news and bad news. Any exuberance about the good news on the level $u_t > 0$ will be dampened by the bad news about the variance due to u_t^2 large. Thus, agents' reactions to large positive shocks and large negative shocks will be asymmetric, which in turn generates asymmetries in endogenous variables. These asymmetries turn out to be more important for the volatilities than for the levels of the variables.

The greater part of the paper will be devoted to studying the *when it rains it pours* mechanism in the context of a consumption CAPM model, of the kind first introduced by Lucas (1978). Despite its well known failings, the consumption CAPM has one important advantage: simplicity. Although the CCAPM is a dynamic general equilibrium model, it is possible to find closed form solutions for some types of asset returns, namely bond returns. By studying these closed form solutions it will be possible to gain some insight into how the *when it rains it pours* mechanism works. Moreover, the CCAPM is flexible enough to generate a wide variety of asset returns, providing an opportunity to compare the workings of the mechanism in bond and equity returns. The final part of the paper is then devoted to a pair of numerical exercises, in order to determine whether the volatility asymmetries generated by the *when it rains it pours* mechanism are empirically relevant and quantitatively significant for reasonable parameter values. It turns out that the degree of countercyclical heteroscedasticity in both bond and equity returns is indeed quantitatively significant and quite similar to that found in the data.

1.1 Asymmetric Volatility

The greater volatility of asset returns during recessions was first noted by Officer (1973). Schwert (1989) presents further evidence that equity and short-term bond returns are more volatile during recessions. In particular, Schwert (1989) reports estimates that monthly equity returns were 68% more volatile during recessions than during expansions in the post-war U.S. data (1953-1987). Over the same period, monthly short-term bond returns were estimated to be 134% more volatile. Such countercyclical heteroscedasticity also seems to be a property of other kinds of economic variables: Schwert (1989) also presents evidence that production growth rates are more volatile during recessions. Although such a wide range of variables displays greater volatility during recessions, theoretical explanations have been specific to equity returns.

For equity returns, two explanations for CCH have been advanced. The most prominent explanation is the "leverage effect", originally due to Black (1976), for which Schwert (1989) provides partial empirical support. During economic contractions, an asset's total value declines, so that the proportion of its value which is levered increases. More highly levered assets are riskier, so the leverage effect leads to equity returns which are more volatile during recessions. However, leverage is not of much help for fully-levered assets, most notably bonds, whose returns also display countercyclical heteroscedasticity. Leverage is of even less help in explaining CCH in more general macroeconomic variables, such as production growth. Thus, it seems that a deeper mechanism is needed, one which is capable of generating asymmetries in volatility over the business cycle in a wider range of variables.

The first objective of this paper is to describe such a deeper mechanism, one which is based upon agents' joint forecasting of levels and variances of relevant driving processes in a dynamic general equilibrium framework. The mechanism is based upon the idea that it is the *sign* of an innovation which determines whether it carries good or bad news on the level, but the *magnitude* which determines whether news on the variance is good or bad. Since sign and magnitude need not coincide, we obtain a richer set of implications for the equilibrium dynamics of endogenous variables. Among these implications is asymmetric volatility in endogenous variables over the business cycle. In particular, it will be shown that greater volatility of endogenous variables results under quite general conditions.

To my knowledge, the only other formal model analyzing a similar mechanism is that of Campbell and Hentschel (1992). They develop an equity-specific *volatility feedback* mechanism which is similar to the *when it rains it pours* mechanism presented in this paper. In the *volatility feedback* mechanism, time-varying second moments also serve to amplify equity returns' reactions to negative innovations in dividends, helping to account for the empirically observed correlation between negative innovations and volatility of equity returns. As the name suggests, it is a feedback mechanism: its focus is upon the effects of current innovations to dividends on current equity return volatility. Further, *volatility feedback* operates within an empirical (non-equilibrium) framework and is equity-specific. In particular, it is based upon a log-linear approximation to the ex definition relationship between returns, prices and dividends, better known as the present-value dividend model of Campbell and Shiller (1988a,b).

In contrast, we are interested in addressing volatility asymmetries in a more general class of variables, and focus on forecasting in dynamic general equilibrium. General equilibrium places stronger restrictions on the relationship between returns, prices and dividends. Not only must the present-value dividend relationship of the *volatility feedback* mechanism continue to hold, but the returns must also be consistent with agents' risk preferences, in conjunction with their expectations on the stochastic dividend process. Given the fundamental importance of agents' risk preferences in determining their reactions to volatility, there is reason to believe that these restrictions may indeed be important, both theoretically and quantitatively. Moreover, *when it rains it pours* differs from *volatility feedback* in its timing. While Campbell and Hentschel (1992) stress the simultaneous effects of innovations, the focus in this paper is on the effects of innovations on agents' *forecasting*, and thus upon future asset return volatility. In this sense, the two approaches are complementary, and will turn out to deliver complementary results.

1.2 GARCH Processes

Clearly, the *when it rains it pours* mechanism depends crucially on the use of squared residuals u_t^2 in variance forecasting. The most direct way - but not the only way - to induce agents to use squared residuals in their optimal forecasts is to assume that innovations are governed by ARCH or GARCH processes.[1] The ARCH specification, introduced by Engle (1982) and generalized by Bollerslev (1986) to GARCH, have been some of the most popular approaches to modelling time-varying second moments.[2]

Literally hundreds of papers have documented the empirical success of GARCH specifications, especially in modelling volatility in financial markets. (For a survey see Bollerslev, Chou and Kroner (1992).) GARCH has also been employed quite widely in modeling the variance of non-financial variables. For example, Gallant and Tauchen (1989) find evidence of GARCH-type volatility in the aggregate consumption process, while Engle (1982) finds evidence of ARCH in inflation. These latter two findings motivate the assumption in this paper that the variances of consumption growth and/or inflation follow an ARCH(1) process.[3] That is, since (G)ARCH specifications do very well at representing the

[1] Note, however, that ARCH or GARCH volatility are not necessary ingredients for asymmetries to arise. As will be seen in Section 6, the crucial point is that the endogenous variable contain a quadratic term. Although (G)ARCH is one particularly elegant way of introducing quadratic terms, it is by no means the only way.

[2] In an ARCH(q) process, next period's conditional variance σ_{t+1}^2 is a linear and stochastic function of q lagged squared residuals $\left(u_t^2, u_{t-1}^2, ...u_{t-q}^2\right)$. The GARCH(p,q) specification adds linear dependence on p previous variances $\left(\sigma_t^2, \sigma_{t-1}^2, ...\sigma_{t-p}^2\right)$. Such a specification is said to include p GARCH terms and q ARCH terms.

[3] The theoretical part of the paper will assume, for the sake of tractability, that consumption

empirical properties of variance processes, it seems natural to integrate them into *theoretical* models as well.

In contrast to the voluminous body of empirical literature, relatively few theoretical models take time-varying second-moments into account. Notable exceptions are Kandel and Stambaugh (1990), Canova and Marrinan (1991,1993), and Bollerslev, Engle and Wooldridge (1988). Canova and Marrinan introduce time varying-volatility by means of GARCH innovations to the money supply and government expenditure functions into an ICCAP model, which they then use to study exchange rate volatility and the term structure of interest rates. Bollerslev, Engle and Wooldridge (1988) introduce time-varying covariances into a CAPM model. Closest in approach to the present model is the work of Kandel and Stambaugh (1990): they introduce second moments of consumption growth which follow a simple autoregressive process into a consumption CAPM framework, and examine the ability of such models to generate large equity premia. In contrast, here we concentrate on variances rather than levels, and upon (G)ARCH second moments.

1.3 Quantitative Significance

The second objective of this paper is to show that the degree of CCH in real and nominal bond returns and equity returns is indeed quantitatively significant and empirically relevant. If agents have low to moderate levels of relative risk aversion, then the model is able to match quite well the 29% by which short-term bond returns were estimated by Schwert (1989) to be more volatile during recessions over the last century. Moreover, it will become clear that greater volatility of real bond returns during recessions may be due to ARCH volatility in consumption growth, while the greater volatility of nominal bond returns during recessions may be due either to ARCH in inflation or in consumption growth or both.

Equity return results are complementary to those of Campbell and Hentschel (1992). In the *when it rains it pours* framework, next-period equity returns turn out to be more volatile during recession, but only at low to moderate degrees of risk aversion. When parameters are chosen to match U.S. monthly consumption data[4], simulated equity returns turn out to be more than twice as volatile during recessions as during expansions. However, the degree of countercyclical heteroscedasticity in equity returns is decreasing in risk aversion. The behavior of *when it rains it pours* in conjunction with risk aversion turns out to very important in interpreting the results, and will be discussed at length in Section 5.

The remainder of the paper is organized as follows: Section 2 presents the

growth follows an ARCH(1) process. I have, however, also examined a numerical example with a GARCH(1,1) consumption process, which gives qualitatively similar (and somewhat stronger) results. Details are available upon request.

[4] See Section 5 for a discussion of the use of monthly versus quarterly data. All simulations have also been performed for the calibration to quarterly data presented in Appendix B.1. Quarterly results do *not* vary in any significant way, and are available upon request.

general framework and derives equilibrium asset returns with ARCH(1) variance forecasting. Section 3 derives closed form solutions for ARCH(1) bond returns and discusses some general properties of bond returns when variances are time-varying. In Section 4 the focus is upon ARCH(1) bond return *volatility*: we first link volatility to innovations, and then innovations to recessions, in order to examine the relationship between volatility and the business cycle more closely. Section 5 presents simulation results on real and nominal bond and equity returns from both ARCH and constant variance models calibrated to U.S. data. Section 6 discusses extensions of the results to non-financial variables, and Section 7 concludes.

2 CAPM with ARCH(1) Variance

In this section, the goal is to introduce symmetric heteroscedasticity in the driving process into a consumption-CAPM model of the type first described in Lucas (1978). This provides a simple stochastic dynamic general equilibrium framework in which to test the ability of joint variance and level forecasting to generate asymmetries in the volatility of endogenous variables. In consumption-CAPM models, these endogenous variables are the returns on claims to aggregate consumption (equity returns) and the returns to one-period bonds. I will describe how equilibrium asset returns are related to the symmetrically heteroscedastic consumption growth process. This relationship will provide a precise basis for the discussion on asymmetric heteroscedasticity in endogenous variables in the sections to follow.

2.1 Asset Returns in an Exchange Economy

Consider a simple stochastic dynamic general equilibrium model of the type first introduced by Lucas (1978). Agents choose streams of consumption and asset holdings $\{c_t, z_t\}_t$ to maximize the discounted sum of future utilities, given the stochastic process for endowments $\{y_t\}_t$. Formally, they solve

$$\max \sum_{t=0}^{\infty} \delta^t u(c_t) \tag{1}$$

subject to the resource constraint

$$\begin{array}{cccc} y_t + R_{t+1} z_t & = & c_t + z_{t+1} & t = 0, 1, ... \\ & \{y_t\}_t & c_{-1}, z_{-1} & \text{given} \end{array} \tag{2}$$

where R_{t+1} is the gross return on the asset z_t. The asset is in zero net supply, so that in equilibrium $z_t = 0$. The equilibrium solution to this optimization problem takes the form of an Euler equation, which may be written as[5]

$$1 = \delta E_t \left\{ \frac{u'(c_{t+1})}{u'(c_t)} R_{t+1} \right\} \tag{3}$$

[5] This Euler equation for general assets was first derived by Grossman and Shiller (1981).

Assuming power utility $u(c_t) = \frac{1}{1-\gamma} c_t^{1-\gamma}$, the Euler equation may be expressed in terms of the consumption growth rate $x_{t+1} \equiv \frac{c_{t+1}}{c_t}$ as:

$$1 = \delta E_t \left\{ x_{t+1}^{-\gamma} R_{t+1} \right\} \tag{4}$$

where γ represents the coefficient of relative risk aversion.

One can now apply the Euler equation (4) for general asset returns in an Lucas exchange economy to two types of assets. I follow the tradition in the Macrofinance literature, and focus on the returns to equity and to one-period bonds.

2.1.1 Equity Returns

In the CCAPM, equity is defined as a claim to consumption and saving is ruled out, hence in equilibrium the dividend d_t is equal to aggregate (per capita) consumption c_t. Its gross return may be expressed as $R_{t+1} = \frac{p_{t+1}+d_{t+1}}{p_t}$. Substituting into equation (4) yields

$$p_t = \delta E_t \left\{ x_{t+1}^{-\gamma} \left\{ p_{t+1} + d_{t+1} \right\} \right\}$$

In a growing economy, dividends d_t and prices p_t are non-stationary. Under balanced growth, however, these variables grow at the same average rate x_t, so that the price-dividend ratio $\frac{p_t}{d_t}$ is stationary. Thus, it is helpful to write the Euler equation for a claim to consumption in terms of stationary variables as

$$\frac{p_t}{d_t} = \delta E_t \left\{ x_{t+1}^{1-\gamma} \left(\frac{p_{t+1}}{d_{t+1}} + 1 \right) \right\} \tag{5}$$

The sequence of price-dividend ratios $\left\{ \frac{p_t}{d_t} \right\}_t$ satisfying equation (5) may be approximated using the parameterized expectations approach of Marcet and Marshall (1994) [See Appendix.C for details.]. Once one has obtained the equilibrium price-dividend sequence $\left\{ \frac{p_t}{d_t} \right\}_t$, equilibrium returns may be recovered as

$$R_{t+1} = x_{t+1} \frac{p_{t+1}/d_{t+1} + 1}{p_t/d_t} \tag{6}$$

2.1.2 Bond Returns

One-period bonds may be represented as claims to an asset paying a dividend of one unit of the consumption good ($d_t = 1$) which mature at $t+1$ ($p_{t+1} = 0$). Substituting into equation (4) yields the following expression for the price of the one-period bond

$$q_t = \delta E_t \left\{ x_{t+1}^{-\gamma} \right\} \tag{7}$$

Under power utility and log-normally distributed consumption growth rates x_{t+1}, Hansen and Singleton (1983) show that it is possible to find analytical solutions for prices and returns on one-period bonds. Since the gross return on a one-period bond is $R_{t+1} = \frac{1}{q_t}$, and thus $r^f_{t+1} = \log R_{t+1} = -\log q_t$, one may use equation (7) above and write:

$$r^f_{t+1} = -\log\delta + \underbrace{\gamma E_t \log x_{t+1}}_{\text{smoothing term}} - \underbrace{\frac{\gamma^2}{2} var_t \log x_{t+1}}_{\text{precautionary term}} \qquad (8)$$

The smoothing term reflects the fact that agents wish to borrow against future consumption growth, placing upward pressure on interest rates. The precautionary term, on the other hand, captures the effects of consumption volatility. The more volatile the future growth rate, the more agents wish to insure themselves by means of precautionary savings. Increased demand for savings will place upward pressure on the interest rate.

Equations (5) and (8) define a relationship between the expected growth rate of consumption x_{t+1} and the equilibrium returns of equity and one-period bonds. Thus, the precise nature of the process governing the consumption growth rate plays a crucial role in determining the properties of the equilibrium returns. The next section describes the driving process for $\log x_{t+1}$ assumed here, which incorporates time-varying second moments.

2.2 Incorporating AR(1)-ARCH(1) Consumption Growth

It is at this point that the framework here diverges from the canonical Lucas/Mehra-Prescott model. The differing element is the introduction of time-varying second moments in the driving process. In particular, it is now time to incorporate symmetric heteroscedasticity in the innovation u_t by means of an ARCH(1) specification.

To see how this works, suppose that the consumption growth rate $\log x_{t+1} \equiv \log\left(\frac{c_{t+1}}{c_t}\right)$ follows an AR(1) process

$$\log x_{t+1} = c + \rho \log x_t + u_{t+1} \qquad |\rho| < 1 \qquad (9)$$

This is a standard approach to modeling consumption growth in the consumption-based asset pricing literature. Now we incorporate heteroscedasticity in the innovation u_t by means of an ARCH(1) process. Innovations which are $u_t \sim ARCH(1)$ may be described as:

$$\begin{aligned} u_{t+1} &\sim N\left(0, \sigma^2_{t+1}\right) \\ \sigma_{t+1} &= \sqrt{\xi + \alpha u_t^2} \cdot v_{t+1} \qquad \text{where } v_{t+1} \sim i.i.d. N(0,1) \end{aligned}$$

The ARCH(1) specification has the convenient property that the conditional expectation of the variance is linear in the lagged squared innovation u_t^2:

$$E_t \sigma^2_{t+1} = \xi + \alpha u_t^2 \qquad (10)$$

Heteroscedasticity in the innovations induces heteroscedasticity in the consumption growth rates $\log x_{t+1}$. Moreover, this heteroscedasticity is *symmetric* over the business cycle: consumption growth is just as volatile in recessions as in expansions. In particular, the consumption growth rate is symmetrically and conditionally log-normally distributed with moments:

$$E_t \log x_{t+1} = c + \rho \log x_t \tag{11}$$
$$var_t \log x_{t+1} = \xi + \alpha u_t^2 \tag{12}$$

Thus, each innovation u_t has an impact on both the expectation of $\log x_{t+1}$ (via $\log x_t$), and upon its variance. A large negative innovation will cause agents to expect future consumption growth to be low and volatile: *when it rains it pours*. Agents observing a large positive innovation, on the other hand, will expect future consumption growth to be high but volatile. In this latter case, any exuberance about high future growth will be dampened by worries about the economy "overheating" due to increased volatility. Consumption CAPM with ARCH(1) Variance

In the following section, we introduce such symmetric ARCH(1) volatility in the driving consumption growth process into the simple consumption-CAPM model. This provides a simple dynamic general equilibrium framework in which to test the ability of the *when it rains it pours* mechanism to generate asymmetries in the volatility of endogenous variables. In consumption-CAPM models, these endogenous variables are the returns on claims to aggregate consumption (equity returns) and the returns to one-period bonds. I will describe how equilibrium asset returns are related to ARCH(1) consumption growth. This relationship will provide a precise basis for the discussion on asymmetric heteroscedasticity in endogenous variables in the sections to follow.

3 Bond Returns with and without ARCH

Now I wish to analyze more carefully the effects of time-varying volatility and "when it rains it pours" variance forecasting. Since bond returns' closed form solutions lend themselves to such careful analysis, I begin by examining bond returns in some detail. The most natural place to start is by comparing the equilibrium bond returns with and without ARCH. For the AR(1)-ARCH(1) framework introduced above, the return to a one-period bond may be obtained by substituting the conditional moments (11) and (12) into the general equation for the bond return (8) to obtain:

$$r_{t+1}^f = -\log\delta + \underbrace{\gamma\left[c + \rho\log x_t\right]}_{\text{smoothing term}} - \underbrace{\frac{\gamma^2}{2}\left[\xi + \alpha u_t^2\right]}_{\text{precautionary term}} \tag{r_{t+1}^f–ARCH}$$

Similarly, when consumption growth variance is constant and equal to σ_x^2 the bond return may be written as:

$$r_{t+1}^f = -\log\delta + \underbrace{\gamma\left[c + \rho\log x_t\right]}_{\text{smoothing term}} - \underbrace{\frac{\gamma^2}{2}\sigma_x^2}_{\text{precautionary term}} \tag{r_{t+1}^f–no ARCH}$$

Clearly, the smoothing effect will be identical in both the ARCH and no-ARCH cases. Thus, any differences in the properties of ARCH and no-ARCH bond returns must be due to their differing precautionary terms, and thus to their variances.

Indeed, the precautionary effects will most likely differ. This is due to the fact that the ARCH variance forecasts are varying over time. At times, the ARCH variance forecast $\xi + \alpha u_t^2$ will be greater than its constant unconditional mean σ_x^2, placing additional downward pressure on the riskfree rate via a stronger precautionary effect. At other times, however, the ARCH variance forecast will be lower than average: relatively calm times will weaken the precautionary motive to save, placing additional upward pressure on the riskfree rate.

Figure (1) illustrates this variance-based difference between ARCH and no-ARCH bond returns. The solid line plots ARCH bond returns as a function of u_t, while the dashed line represents bond returns when variance is constant. For large magnitude innovations, the news on the variance is bad, placing downward pressure on the risk-free rate via the precautionary motive, and the ARCH bond returns curve lies below the corresponding constant variance line. For small magnitude innovations, however, the good news on the variance puts upward pressure on the risk-free rate. As a result, the ARCH bond return is greater than its constant-variance counterpart, and the ARCH curve lies above the no-ARCH line.

The quantitatively significant effects of time-varying second moments are not, however, on the bond returns themselves nor on their overall volatility. In fact, overall unconditional moments of bond returns in the ARCH and no ARCH cases are almost indistinguishable. Table 3.1 reports sample mean bond returns for the ARCH and no ARCH calibrations for three degrees of risk aversion γ. The greatest difference is one basis point (one-hundredth of a percentage point). Similarly, Table 3.2 below shows that overall bond return volatility is not affected to any great extent by variance-forecasting either. Thus, introducing time-varying second moments to the driving process leads to bond returns whose overall moments are nearly indistinguishable from their constant variance counterparts.

Table 3.1: Average Monthly Bond Returns

γ	1.5	2.0	4.0
ARCH	1.26 %	1.34 %	1.68 %
no ARCH	1.26 %	1.34 %	1.67 %

Table 3.2: Bond Return Volatilities

γ	1.5	2.0	4.0
ARCH	0.10%	0.15 %	0.29 %
no ARCH	0.11 %	0.15 %	0.29 %

Tables 3.1 and 3.2: Results refer to 50 runs of 5000 periods each of ARCH and no ARCH models where the consumption growth process has been chosen to match U.S. monthly data, as detailed in Section 5. It was never possible to reject the null hypothesis that the ARCH and no ARCH sample means were equal.

4 Asymmetric Volatility of ARCH Bond Returns

Although the unconditional moments are scarcely affected by variance-forecasting, variance forecasting *does* have significant and interesting effects on the volatility of bond returns over the business cycle. To see this, take a second look at Figure (1): it illustrates the fact that the ARCH bond return may be expressed as a *quadratic* function of the date t innovation to consumption growth u_t as:

$$r^f_{t+1}(u_t) = k_{A,t-1} + \gamma\rho u_t - \frac{\gamma^2}{2}\alpha u_t^2 \qquad (r^f_{t+1} \text{ - ARCH})$$

$$\frac{dr^f_{t+1}(u_t)}{du_t} = \gamma\rho - \gamma^2\alpha u_t \qquad \left(\frac{dr^f_{t+1}}{du_t} \text{ - ARCH}\right)$$

where all elements of $k_{A,t-1} \equiv -\log\delta + \gamma\left[c(1+\rho) + \rho^2 \log x_{t-1}\right] - \frac{\gamma^2}{2}\xi$ are parameters or constant (already realized) at date t.

In contrast, when variance is constant and equal to σ_x^2, as it is in the canonical model, the bond return may be written as a *linear* function of u_t:

$$r^f_{t+1} = k_{nA,t-1} + \gamma\rho u_t \qquad (r^f_{t+1} \text{ - no ARCH})$$

$$\frac{dr^f_{t+1}}{du_t} = \gamma\rho \qquad (r^f_{t+1} \text{ - no ARCH})$$

where $k_{nA,t-1} = -\log\delta + \gamma\left[c(1+\rho) + \rho^2 \log x_{t-1}\right] - \frac{\gamma^2}{2}\sigma_x^2$.

What does this mean for the reaction of bond returns to positive and negative innovations? Note that for the concave bond return function (13), the slope of the bond return is decreasing in u_t, while for the linear no-ARCH bond return line, the slope is constant. This is illustrated in Figure (2): in the no-ARCH case, the constant slope translates into bond return which react symmetrically to positive and negative innovations. In the ARCH case, however, the decreasing slope translates into stronger reactions to negative innovations than to positive ones of equal magnitude. This is reflected by the fact that the ARCH plot of $\frac{dr^f_{t+1}}{du_t}$ lies above the horizontal no-ARCH line for all negative innovations, while it lies below the horizontal no-ARCH line whenever $u_t > 0$. *Thus, variance-forecasting tends to amplify the effects of negative shocks, while dampening the effects of positive shocks.* This clearly induces a negative correlation between innovations to the consumption growth rate process and bond return volatility, which is considered in greater detail in what follows.

4.1 Negatively Skewed Heteroscedasticity

Negatively Skewed Heteroscedasticity (NSH) captures the idea that negative innovations are associated with greater volatility. Formally, NSH may be expressed by means of conditional bond return variance[6] which is greater conditional on innovations being negative:

$$\underbrace{E\left\{\left(r_{t+1}^{f}\left(u_t\right)-E_{t-1}r_{t+1}^{f}\right)^2|u_t<0\right\}}_{var_t\left[r_{t+1}^{f}|u_t<0\right]}>\underbrace{E\left\{\left(r_{t+1}^{f}\left(u_t\right)-E_{t-1}r_{t+1}^{f}\right)^2|u_t\geq 0\right\}}_{var_t\left[r_{t+1}^{f}|u_t\geq 0\right]} \tag{13}$$

Equivalently, one may write the above equation in terms of absolute deviations from the conditional mean as:

$$E\left\{\left|r_{t+1}^{f}\left(u_t\right)-E_{t-1}r_{t+1}^{f}\right||u_t<0\right\}>E\left\{\left|r_{t+1}^{f}\left(u_t\right)-E_{t-1}r_{t+1}^{f}\right||u_t\geq 0\right\} \tag{14}$$

4.1.1 No ARCH, no NSH

First, it can be shown that constant-variance bond returns do *not* exhibit skewed heteroscedasticity. That is, in the no-ARCH case symmetrically time-varying second moments to the driving process lead to symmetrically time-varying second moments in the endogenous variable.

Begin by noting that in the constant variance model the innovation needed to induce the conditional mean bond return is $\widetilde{u}_t = 0$. That is $\widetilde{u}_t = 0$ is the innovation which satisfies $E_{t-1}r^f = r_{t+1}^{f}\left(\widetilde{u}_t\right)$.[7] For constant-variance bond returns, then, the absolute deviation in the bond return from its conditional mean due to an innovation u_t may be written as

$$s^{NA}\left(u_t\right)\equiv\left|r_{t+1}^{f}\left(u_t\right)-E_{t-1}r_{t+1}^{f}\right|=\left|r_{t+1}^{f}\left(u_t\right)-r_{t+1}^{f}\left(0\right)\right|$$

Thus, $s^{NA}\left(u_t\right)$ may also be expressed as the integral:

$$s^{NA}\left(u_t\right)=\left|\int_0^{u_t}\frac{dr_{t+1}^{f}\left(u\right)}{du}du\right| \tag{15}$$

[6] Note that although the bond return between dates t and $t+1$ is called r_{t+1}^{f}, this bond return is determined at date t based upon date t expectations on date $t+1$ growth rates. Thus, the appropriate conditional variance concept is the expectation of the squared difference between the bond return's realization (at date t) and its conditional expectation at date $t-1$. Formally: $var_t\left\{r_{t+1}^{f}\right\}=E\left\{\left(r_{t+1}^{f}\left(u_t\right)-E_{t-1}r_{t+1}^{f}\right)^2\right\}$.

[7] Also, $\widetilde{u}_t = 0$ also induces the unconditional mean bond return on average in the constant variance model. See Appendix 1.A for details.

The integral (15) yields a particularly convenient graphical representation of $s^{NA}(u_t)$. In Figure (3), one can see that (15) corresponds to the shaded area under the horizontal no-ARCH line between u_t and 0.

To show that bond return volatility is symmetric in the constant variance case, it is necessary to show that volatility is equally large conditional on innovations being negative or positive. Thus, one must show that the average size of the shaded area taken only over negative innovations $u_t < 0$ is just as large as the average size of the shaded area taken only over positive innovations. This is clearly the case. To see this, note that for each pair of positive and negative innovations of equal magnitude (u^+, u^-), the absolute deviations are also equal, as illustrated in Figure (4). Formally:

$$s^{nA}(u+) = s^{nA}(u-) \qquad \text{for} \quad \begin{matrix} u^+ > 0 \\ u^- = -u^+ \end{matrix} \tag{16}$$

Since u_t is symmetrically distributed about zero, the equal magnitude positive and negative innovations u^+ and u^- are equally likely. Thus, it is easy to see that

$$E\left\{s^{nA}(u_t)\,|u_t < 0\right\} = E\left\{s^{nA}(u_t)\,|u_t \geq 0\right\}$$

In the no-ARCH case, positive and negative innovations have *symmetric* effects not only on the bond returns themselves, but also upon their deviations from their conditional means $E_{t-1}r^f_{t+1}$, and thus on their conditional volatility.

4.1.2 ARCH and NSH

The ARCH case is somewhat more complex. First consider something close to the absolute deviation of the bond return from its conditional mean, namely the deviation from its $u_t = 0$ value:

$$s_0^A(u_t) = \left| r^f_{t+1}(u_t) - r^f_{t+1}(0) \right|$$

It is easy to see that $s_0^A(u_t)$ corresponds to the vertically striped region under the ARCH line between u_t and 0 in Figure (5). It is also clear that the ARCH deviation will be of greater magnitude for negative innovations. The area under the ARCH line between 0 and u^- is clearly greater than that between 0 and u^+ in Figure (6) so that

$$s_0^A\left(u^-\right) > s_0^A(u+) \qquad \text{for} \quad \begin{matrix} u^+ > 0 \\ u^- = -u^+ \end{matrix}$$

Now symmetry of the distribution of u_t about zero implies that the average size of the shaded area under the ARCH line will be greater over all negative shocks:

$$E\left\{s_0^A(u_t)\,|u_t < 0\right\} > E\left\{s_0^A(u_t)\,|u_t \geq 0\right\} \tag{17}$$

Thus, in the ARCH case, negative innovations have greater effects on s_0^A than do positive innovations, inducing an asymmetry in the reaction of bond returns to positive and negative innovations *if the conditional mean-inducing innovation were* $\widetilde{u}_t = 0$.

However, (17) is not necessarily equivalent to asymmetric volatility for the ARCH case. This is due to the fact that in the ARCH case, $u_t = 0$ is *not* the innovation required to induce the conditional (or unconditional) mean bond return $E_{t-1}r^f_{t+1}$ (or Er^f). By the two lemmas below, whether the (conditional) mean-inducing innovation is positive or negative will depend upon the serial correlation of consumption growth ρ. In what follows, the focus will be on the empirically relevant case of positive serial correlation in growth rates. Positive serial correlation in $\log x_t$ leads to mean-inducing innovations which are positive, which turns out to work in favor or NSH.[8]

Lemma

Assume that $\gamma, \alpha, \xi > 0$, *so that agents are risk averse, and variances are positively serially correlated and guaranteed to be positive. For* $\rho > 0$, *the conditional mean-inducing innovation* $\widetilde{u}^c_t$ *is given by*

$$\widetilde{u}^c_t = \frac{-\rho + \sqrt{\rho^2 + \gamma^2 \alpha \cdot c\left(u^2_{t-1}\right)}}{\gamma \cdot c\left(u^2_{t-1}\right)} > 0$$

for all values of u^2_{t-1} *where* $c\left(u^2_{t-1}\right) = \equiv \left(\rho^2 + \alpha\right)\left(\xi + \alpha u^2_{t-1}\right)$.

Proof

See Appendix A.■

Positive $\widetilde{u}_t$ works in favor of NSH. The reason is that any given negative innovation will be further away from to a positive $\widetilde{u}_t$ than from 0. This works *augments* the asymmetry due to the negatively sloped ARCH $\frac{dr^f}{du}$ line.

To see this more clearly, recall that $s^A(u_t)$ may be expressed as the integral:

$$s^A(u_t) = \left| \int_{\widetilde{u}_t}^{u_t} \frac{dr^f_{t+1}(u)}{du} du \right| \tag{18}$$

For $\widetilde{u}_t = 0$, the integrals for positive and negative shocks u^+ and u^- are taken over equally sized ranges. Thus, the fact that the integrand is always greater for u^- is sufficient for the u^- -integral to be greater, so that NSH holds. For $\widetilde{u}_t > 0$, negative innovations must travel even further to reach $\widetilde{u}_t$, so that the u^-- integral is taken over a greater range than the corresponding u^+-integral, reinforcing NSH. If $\widetilde{u}_t$ were negative, however, negative innovations would not have to "travel" as far to reach $\widetilde{u}_t$ as do positive ones. Thus, the range of the u^- integral would be smaller, while its integrand is greater, so that it would not certain which of these countervailing influences on $s^A(u_t)$ will prevail.

[8] When growth is negatively serially correlated ($\rho < 0$), however, the (conditional) mean-inducing innovation is negative. A proof is given in Appendix 1.A.

Table 4.1: Mean-Inducing Innovations and Asymmetric Absolute Deviations

Case	Range Effect	Integrand Effect	Total Effect
$\widetilde{u}_t < 0$	$\lvert\widetilde{u}_t - u^-\rvert < \lvert\widetilde{u}_t - u^+\rvert$	$\frac{dr^f_{t+1}(u^-)}{du} > \frac{dr^f_{t+1}(u^+)}{du}$	ambiguous
$\widetilde{u}_t = 0$	$\lvert\widetilde{u}_t - u^-\rvert = \lvert\widetilde{u}_t - u^+\rvert$	$\frac{dr^f_{t+1}(u^-)}{du} > \frac{dr^f_{t+1}(u^+)}{du}$	$s^A(u^-) > s^A(u^+)$
$\widetilde{u}_t > 0$	$\lvert\widetilde{u}_t - u^-\rvert > \lvert\widetilde{u}_t - u^+\rvert$	$\frac{dr^f_{t+1}(u^-)}{du} > \frac{dr^f_{t+1}(u^+)}{du}$	$s^A(u^-) > s^A(u^+)$

Taken together, the greater is the slope of the ARCH-line in Figure (6) and the larger is the (positive) mean-inducing innovation $\widetilde{u}_t$, the more pronounced will be the negative skewness of the heteroscedasticity.[9]

4.2 Greater Volatility during Recessions

What is really of interest here, however, is the relationship between *recessions* and bond return volatility. Bond returns which are countercyclically heteroscedastic (CCH) display conditional variances which are greater whenever the innovation u_t is recessionary. Formally, CCH is said to hold whenever

$$E\left\{s^A(u_t)\,|u_t \text{ recessionary}\right\} > E\left\{s^A(u_t)\,|u_t \text{ expansionary}\right\}$$

To the extent that negative innovations to consumption growth are linked to recessions, one would expect NSH to be linked with CCH. It turns out that in growing economies, CCH is more likely to hold than NSH. Further, CCH may even hold in an economy which is in long-run decline. In order to examine this more closely, it is first necessary to define more precisely what is meant by recession and expansion.

4.2.1 Recession and Expansion

An innovation u_t is called *recessionary* whenever it causes the growth rate $\log x_t$ to be negative. More precisely: u_t is recessionary whenever $\log x_t = c + \rho \log x_{t-1} + u_t \leq 0$, which translates into a condition on u_t as

$$u_t \in \{U_t^{rec}\} \quad \text{whenever} \quad u_t \leq -c - \rho \log x_{t-1} \equiv \overline{u}_t \tag{19}$$

$$u_t \in \{U_t^{\exp}\} \quad \text{whenever} \quad u_t > -c - \rho \log x_{t-1} \equiv \overline{u}_t \tag{20}$$

Thus, for any given $\log x_{t-1}$ and any AR(1) parameters, the recessionary threshold $\overline{u}_t$ divides the support of innovations into two disjoint subsets. This is illustrated in Figure (7). The subset of recessionary innovations U_t^{rec} is that subset of the real numbers which is lies below the recessionary threshold, while the subset of expansionary innovations $U_t^{\exp}$ is its complement.

Note also that the recessionary threshold $\overline{u}_t$ will be shifting over time depending upon last period's growth rate $\log x_{t-1}$. If last period's growth rate

[9] By completely analogous arguments, it is possible to see that bond return volatility exhibits positively skewed heterskedasticity when growth is negatively serially correlated ($\rho < 0$).

was large and positive, then it will take a relatively large negative innovation to throw the economy into recession. If, however, last period's growth rate $\log x_{t-1}$ was already recessionary, then it is possible that even small positive innovations will be sufficient to keep the economy in recession. From equation (19) one obtains that the recessionary threshold will be symmetrically and normally distributed with mean $-\mu_x$ and variance $\rho^2\sigma_x^2$. Thus, the greater the growth trend in the economy, the more strongly negative will the recessionary threshold tend to be.

4.2.2 Countercyclical Heteroscedasticity in a Growing Economy

Now it is possible to make more precise the idea that bond returns are more volatile during recessions than expansions. Formally, consider the CCH property for given $\widetilde{u}_t$ and given recessionary threshold $\overline{u}_t$: CCH holds for $(\widetilde{u}_t, \overline{u}_t)$ whenever

$$E\left\{s_{t+1}^A(u_t) \mid u_t < \overline{u}_t; \widetilde{u}_t\right\} > E\left\{s_{t+1}^A(u_t) \mid u_t > \overline{u}_t; \widetilde{u}_t\right\} \tag{21}$$

The absolute deviation in the bond return at date $t+1$ may be written as a function of the innovation at date t as

$$s_{t+1}^A = \left|r_{t+1}^f(u_t) - r_{t+1}^f(\widetilde{u}_t)\right| = \left|\int_{\widetilde{u}_t}^{u_t} \frac{dr_{t+1}^f(u)}{du} du\right| \tag{22}$$

Proposition

In a CCAPM economy with $\gamma, \rho, \alpha, \xi > 0$, so that agents are risk averse, endowment growth and volatility are both positively serially correlated, and volatility is guaranteed to be positive, CCH holds all pairs $(\overline{u}_t, \widetilde{u}_t)$ such that $\overline{u}_t \leq \widetilde{u}_t$.

Proof

Since $\gamma, \alpha, \xi > 0$, Lemma 1 implies that the (conditional) mean-inducing innovation $\widetilde{u}_t$ is positive for $\rho > 0$. Further, $\gamma, \rho, \alpha > 0$ guarantees that the integrand $\frac{dr_{t+1}^f(u_t)}{du_t}$ is monotonically decreasing in u_t.

Thus, once again $s_{t+1}^A(u^-) > s_{t+1}^A(u^+)$ for all pairs (u^-, u^+) where $-u^- = u^+ > 0$, both due to the larger range $[u^-, \widetilde{u}_t]$ when $\widetilde{u}_t > 0$, and due to the larger integrand of $s_{t+1}^A(u^-)$. Since the economy is growing, $\mu_x > 0$, and the recessionary threshold $\overline{u}_t$ is symmetrically distributed about $-\mu_x \leq 0$.

First, consider the case that the recessionary threshold is negative, so that $\overline{u}_t < 0$. First note that $s_{t+1}^A(u^-) > s_{t+1}^A(u^+)$ clearly also holds for each member of the subset of all pairs (u^-, u^+) such that $u^- \in U_t^{rec}$. Formally:

$$E\left\{s_{t+1}^A(u_t) | u_t \leq \overline{u}_t\right\} > E\left\{s_{t+1}^A(u_t) | u_t \geq -\overline{u}_t > 0\right\} \tag{23}$$

Furthermore, $s_{t+1}^A(u^-)$ is monotonically increasing in the magnitude of u^-. Thus, the deviation due to any negative and recessionary innovation is larger than that due to any negative but expansionary innovation. Formally, this implies that :

$$E\left\{s_{t+1}^A(u_t) | u_t \leq \overline{u}_t\right\} > E\left\{s_{t+1}^A(u_t) | \overline{u}_t < u_t \leq 0\right\} \tag{24}$$

Taken together, (23) and (24) imply that bond returns satisfy CCH for all $(\overline{u}_t, \widetilde{u}_t)$ *such that* $\overline{u}_t < 0 < \widetilde{u}_t$.

Now consider the case that the recessionary threshold $\overline{u}_t$ *is positive but less than the mean-inducing innovation* $\widetilde{u}_t$, *so that* $0 < \overline{u}_t < \widetilde{u}_t$. *Let* $-u^- = u^+ > 0$. *By monotonicity of* $\frac{dr^f_{t+1}(u_t)}{du_t}$, *recall that* $s^A_{t+1}(u^-) > s^A_{t+1}(u^+)$ *for all pairs* (u^-, u^+). *This is also true for the subset of all large magnitude innovations* $|u_t| > \widetilde{u}_t$, *and innovations are symmetrically distributed about zero, so that*

$$E\left\{s^A_{t+1}(u_t)\,|u_t \le -\widetilde{u}_t\right\} > E\left\{s^A_{t+1}(u_t)\,|u_t \ge \widetilde{u}_t\right\} \tag{25}$$

Now consider the small magnitude innovations $|u_t| < \widetilde{u}_t$. $s^A_{t+1}(u_t)$ *is monotonically decreasing over the interval* $|u_t| < \widetilde{u}_t$, *so that taking expectations yields*

$$E\left\{s^A_{t+1}(u_t)\,| - \widetilde{u}_t < u_t \le \overline{u}_t\right\} > E\left\{s^A_{t+1}(u_t)\,|\overline{u}_t \le u_t < \widetilde{u}_t\right\} \tag{26}$$

Taken together, (25) and (26) imply that

$$E\left\{s^A_{t+1}(u_t)\,|u_t \le \overline{u}_t\right\} > E\left\{s^A_{t+1}(u_t)\,|u_t \ge \overline{u}_t\right\} \tag{27}$$

or equivalently $E\left\{s^A_{t+1}(u_t)\,|u_t \in U^{rec}_t\right\} > E\left\{s^A_{t+1}(u_t)\,|u_t \in U^{\exp}_t\right\}$, *so that CCH holds for all* $(\overline{u}_t, \widetilde{u}_t)$ *such that* $0 < \overline{u}_t < \widetilde{u}_t$.■

Now, we proceed to define what it means for bonds to be countercyclically heteroscedastic overall. CCH is said to hold overall if it also holds when expectations are taken over the entire distribution of recessionary thresholds $\overline{u}_t$.

Proposition

In a growing CCAPM economy with $\gamma, \rho, \alpha, \xi > 0$, *so that agents are risk averse, endowment growth and volatility are both positively serially correlated, and volatility is guaranteed to be positive, CCH holds overall..*

Proof

Since $\gamma, \alpha, \xi > 0$, *Lemma 1 implies that the (conditional) mean-inducing innovation* $\widetilde{u}_t$ *is positive for* $\rho > 0$. *Further,* $\gamma, \rho, \alpha > 0$ *guarantees that the integrand* $\frac{dr^f_{t+1}(u_t)}{du_t}$ *is monotonically decreasing in* u_t.

Thus, once again $s^A_{t+1}(u^-) > s^A_{t+1}(u^+)$ *for all pairs* (u^-, u^+) *where* $-u^- = u^+ > 0$, *both due to the larger range* $[u^-, \widetilde{u}_t]$ *when* $\widetilde{u}_t > 0$, *and due to the larger integrand of* $s^A_{t+1}(u^-)$. *Since the economy is growing,* $\mu_x > 0$, *and the recessionary threshold* $\overline{u}_t$ *is symmetrically distributed about* $-\mu_x \le 0$, *and* $E\overline{u}_t \le 0 \le \widetilde{u}_t$.

By Proposition 1, CCH holds for all pairs $(\overline{u}_t, \widetilde{u}_t)$ *such that* $\overline{u}_t \le \widetilde{u}_t$, *and thus for that part of the distribution of* $\overline{u}_t$ *which lies below* $\widetilde{u}_t \ge 0$.

Now, consider the case that the recessionary threshold $\overline{u}_t$ *is positive and greater than the mean-inducing innovation, so that* $\overline{u}_t > \widetilde{u}_t > 0$. *Recall that for a growing economy,* $\overline{u}_t$ *is distributed symmetrically about* $-\mu_x < 0$, *so that for each positive recessionary threshold* $\overline{u}_t > 0$, *there as an equal magnitude negative one* $-(\overline{u}_t + \varepsilon) < 0$ *which is just as likely (where* $\varepsilon > 0$). *Consider such a pair of thresholds* $(-(\overline{u}_t + \varepsilon), \overline{u}_t)$. *Begin by considering the large magnitude innovations. Note that for each pair* $(-\overline{u}_t, \overline{u}_t)$ *of recessionary thresholds, all*

$u_t \leq -(\overline{u}_t + \varepsilon)$ *are recessionary, while all* $u_t \geq \overline{u}_t$ *are expansionary. Further, the expected value of the deviation taken over all such large magnitude recessionary innovations will be greater than that taken all large magnitude expansionary innovations. Formally,* $\widetilde{u}_t > 0$ *and* $\frac{dr^f_{t+1}(u_t)}{du_t}$ *monotonically decreasing guarantee that*

$$E\left\{ s^A_{t+1}(u_t) \,|\underbrace{u_t \leq -(\overline{u}_t + \varepsilon)}_{\text{recessionary}} \right\} \quad > \quad E\left\{ s^A_{t+1}(u_t) \,|\, \underbrace{u_t \geq \overline{u}_t}_{\text{expansionary}} \right\} \qquad (28)$$

$$\text{for each pair } \left(-(\overline{u}_t + \varepsilon), \overline{u}_t\right)$$

Now, consider small magnitude innovations $-(\overline{u}_t + \varepsilon) < u_t < \overline{u}_t$*. Define the expected deviation taken over all small magnitude innovations as* $E\left\{s^A_{t+1}(u_t) \,|-(\overline{u}_t + \varepsilon) < u_t < \overline{u}_t\right\} = M(\overline{u}_t)$*. Recall that* $-(\overline{u}_t + \varepsilon)$ *is at least as likely to be the recessionary threshold as is* $\overline{u}_t$*, so that these small magnitude innovations are at least as likely to be recessionary as expansionary in a growing economy. Formally for each pair of equally likely recessionary thresholds* $(-(\overline{u}_t + \varepsilon), \overline{u}_t)$ *:*

$$prob\{u_t \in U^{rec}_t\} = prob\{u_t \in U^{\exp}_t\} = \frac{1}{2} \text{ for all } -(\overline{u}_t - \varepsilon) < u_t < \overline{u}_t \qquad (29)$$

Taken together, (28) and (29) imply that for each pair of recessionary thresholds $\left(-(\overline{u}_t + \varepsilon), \overline{u}_t\right)$

$$\begin{aligned} E\left\{s^A_{t+1}(u_t)\,|u_t \in U^{rec}_t\right\} &= E\left\{s^A_{t+1}(u_t)\,|u_t \leq -\overline{u}_t\right\} + \frac{1}{2}M(\overline{u}_t) \\ &> E\left\{s^A_{t+1}(u_t)\,|u_t \geq \overline{u}_t\right\} + \frac{1}{2}M(\overline{u}_t) \\ &= E\left\{s^A_{t+1}(u_t)\,|u_t \in U^{\exp}_t\right\} \end{aligned}$$

Summing over all pairs of recessionary thresholds such that $\overline{u}_t > \widetilde{u}_t$ *then yields that CCH holds for that part of the distribution of* $\overline{u}_t$ *which lies above* $\widetilde{u}_t$ *as well.*■

Thus, CCH holds over the distribution of recessionary thresholds as long as $\overline{u}_t$ is distributed symmetrically about some negative value. The factors favoring CCH overall are high mean consumption growth rates μ_x, as well as positive coefficients of risk aversion γ, and positive serial correlations in volatility α and growth rates ρ. Risk aversion and serial correlation in volatility work together to induce the asymmetric reactions to positive and negative innovations in the first place, via the slope effect. Since the magnitude of the (positive) mean-inducing innovation is increasing in ρ, greater degrees of serial correlation in growth rates favors CCH by increasing the size of the range over which the integral in equation (22) is taken. The simulation results will confirm that the degree of CCH found in the model is indeed increasing in each of these factors.

5 Simulation Results

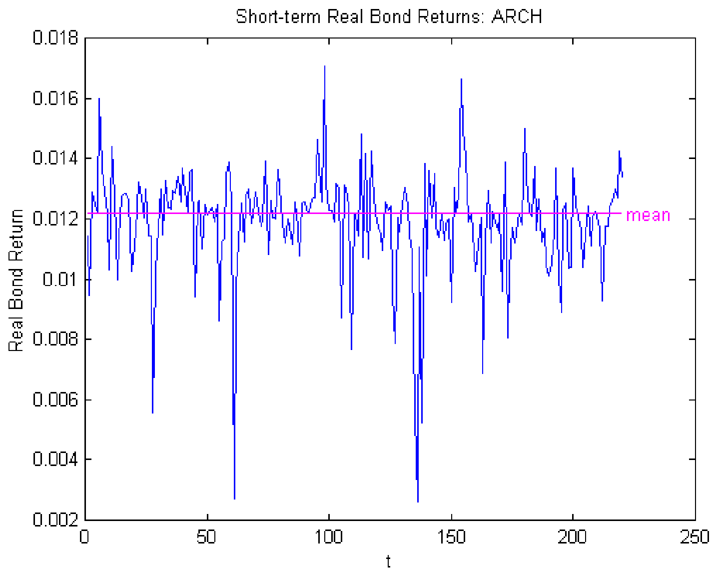
In order to examine whether the degree of cyclical variation in volatilities is quantitatively significant and on the order of magnitude of the empirically observed values we perform three numerical exercises. First, we stick close to the theoretical results of the previous section, and focus on real bond returns. The real bond return simulations confirm the theoretical analysis: the increase in volatility during recessions is sharply increasing in the amount of ARCH present in real consumption growth. Also, it is shown to be essential that the economy be growing, as the theoretical analysis had indicated.

Most available empirical evidence, however, focuses on *nominal* bond returns. Accordingly, the main quantitative results of this paper focus on nominal bond returns as well. To this end, the ARCH model is extended to nominal bonds. It will turn out to be the case that inflation and consumption growth play analogous roles in the nominal bond return. Thus, in the extended model, either ARCH in consumption growth or ARCH in inflation or both could lead to nominal bond returns which are more volatile during recessions. However, due to the role of the inflation risk premium, it will become essential that risk aversion not be too high. This is borne out in the subsequent numerical example, calibrated to match US post-war data on real consumption growth and inflation. Although no significant degree of ARCH is found in monthly per capita real consumption growth, significant ARCH effects are found in the monthly inflation series. The simulation results show that the ARCH in U.S. post-war inflation is also capable of generating significantly greater volatility in nominal bond returns. Moreover, the increase in volatility obtained matches well that estimated by Schwert for U.S. data.

Finally, a third numerical exercise is performed to check the ARCH model's ability to account for increased volatility in equity returns during recessions. It turns out that only moderate degrees of ARCH in dividend growth are necessary to generate significant increases in equity return volatility during recessions.

5.1 Countercyclical Heteroscedasticity in Real Bonds

Indeed, numerical simulations confirm that the increase in real bond return volatility during recessions due to the *when it rains it pours* mechanism is significant. Figures 1 and 2 below provide striking visual evidence. Clearly, the ARCH-driven real bond returns of upper figure tend to display greater downward reactions, reflecting their greater volatility during low or negative growth periods. In contrast, the constant variance real bond returns of the lower figure show balanced upward and downward reactions, reflecting the symmetric effects of positive and negative innovations to the growth rate.



Short-term real bond returns when log consumption growth follows and AR(1)-ARCH(1) process. Parameters: $\gamma = 2.0$, $\alpha = 0.20$, $\rho = 0.24$.

Short-term real bond returns when log consumption growth follows a constant variance process. Parameters: $\gamma = 2.0$, $\rho = 0.24$.

5.1.1 Measuring the increase in volatility during recessions

As convincing as the above figures may be, the increase in volatility in the simulated data must be measured. To compare volatility in recessions and expansions, I follow Schwert (1989) in regressing the absolute deviation of the bond return from its conditional mean $\left|r^f_{t+1}(u_t) - E_{t-1}r^f_{t+1}\right|$ on a constant and a contraction dummy variable. $Contr_t$ takes on the value 1 whenever the innovation at t leads to a recession $u_t \in U^{rec}_t$, and the value 0 otherwise:

$$\left|r^f_{t+1}(u_t) - E_{t-1}r^f_{t+1}\right| = \beta_1 + \beta_2 \cdot contr_t + \varepsilon_t \tag{30}$$

Clearly, an estimate $\widehat{\beta}_2$ which is positive and significant implies that volatility of the bond return is greater during recessions. Further, the ratio $\widehat{\beta}_2/\widehat{\beta}_1$ provides a measure of the percentage increase in volatility during recessions over expansions. It is this measure which will be referred to as the "increase in volatility during recessions" in what follows.

5.1.2 Parameterization to match U.S. post-war data

In order to obtain numerical values for real asset returns one needs to choose values for six parameters: the unconditional mean and standard deviation of log consumption growth μ_x and σ_x, serial correlation ρ, serial correlation in the variances (the ARCH parameter α), risk aversion γ, and the discount factor δ. First, the unconditional moments of the AR(1) process are chosen to match the unconditional moments of U.S. monthly per capital growth in real consumption of non-durables and services. Using monthly NIPA data from the Bureau of Economic Analysis (available from 1967:01 to 2000:09), μ_x is set to 0.1075%, and σ_x to 0.3951%. Risk aversion is chosen to take the values $\gamma \in \{1.0, 2.0, 4.0\}$. These choices are quite conservative, and are well within the range generally considered in the macrofinance literature.The discount factor δ is set to 0.99 : it is easy to see that the discount factor is only a scaling parameter with no effect on variances, so that there is no point in varying it

The serial correlations ρ and α, however, are more difficult to pin down. Wilcox (1992) argues that monthly consumption data have dubious time series properties: the negative serial correlation measured in monthly consumption data is likely to be caused by measurement error. Further, these measurement problems make it impossible to test reliably for the presence of ARCH in monthly consumption data. These rather significant calibration problems with real data are addressed in two complementary ways: by checking the robustness of the real results to the questionable parameters and by shifting the focus to more reliably measured data. In the first approach, ρ and α are varied widely in a sensitivity analysis, and then reasonable parameters are found which generate results which are consistent with the data. In the second approach, the real consumption problems will be addressed by shifting the focus to *nominal* data, and thus more reliably measured inflation series. *It is important to emphasize that the main quantitative results of this paper do not depend upon the properties* *of the unreliable consumption series, but upon the presence of ARCH in more reliably-measured inflation series.*[10]

5.1.3 How much ARCH in consumption is necessary?

The ARCH(1) model can indeed account for significant degrees of greater real bond return volatility during recession. In order to gauge which kind of consumption process would be capable of generating increases in volatility which are consistent with those of the data, I present more detailed results for that grid point which is closest to the long data estimates of Schwert (1989).

Table 5.1: Countercyclical Heteroscedasticity in Bond Returns ARCH(1) versus no ARCH

γ	$\%\Delta s$ constant variance	$\%\Delta s$ ARCH(1)
1.0	1 %	27 %
2.0	1 %	27 %
4.0	1 %	28 %
data		
1859-1987		29 %
1953-1987		134 %

Results are summarized in Table 5.1. The table compare the percentage increase in volatility during recession $\%\Delta s$ in the canonical constant variance and the ARCH(1) models when $\alpha = 0.20$. Clearly, the constant variance model is *not* able to account for any significant increase in volatility of bond returns during recessions. Detailed simulation results for the constant variance model with $\rho = 0.24$ are presented in Table 5.3.[11] γ gives the coefficient of relative risk aversion, while the first two columns represent OLS estimates of the coefficients of regression equation (30), with corresponding White corrected-for-heteroscedasticity t-values in brackets below. $\widehat{\beta}_2$ is slightly positive but not significant, indicating that *there is no significant increase in volatility during recession.* (Indeed, no combination of parameters was able to generate significant increases in volatility in the constant variance model.)

In contrast, the ARCH(1) model with $\alpha = 0.20$ matches the estimates from the data quite well. $\widehat{\beta}_2$ is positive and highly significant, reflecting a significant increase in volatility during recession. At all levels of risk aversion γ, the increase in volatility during recessions is within 2% of the data estimates. Thus, only moderate level of ARCH are necessary to generate volatility asymmetries which are in line with those found in the data. Moreover, as can be seen from the t-values in Table 5.2, the increases in volatility are highly significant. In fact,

[10] In fact, the paper which laid the theoretical foundations for ARCH [Engle (1982)] was concerned with estimating inflation. Thus, the presence of ARCH in inflation data has been well-known and accepted for decades.

[11] All simulation results are averages over 50 runs of 5000 periods each.

the increases in volatility were significant for all combinations of α and ρ, except those where both α and ρ simultaneously take their lowest magnitudes of 0.0 to 0.20 and 0.0, respectively.

Table 5.2: Bond Return Volatility in Recession and Expansion ARCH(1) model

	$s^{\exp}$	$s^{rec}-s^{\exp}$	s^{rec}	$\%\Delta s$
γ	$\widehat{\beta}_1$	$\widehat{\beta}_2$	$\widehat{\beta}_1+\widehat{\beta}_2$	$\widehat{\beta}_2/\widehat{\beta}_1$
1.0	6.99×10^{-4}	1.90×10^{-4}	8.89×10^{-4}	27 %
	[42.47]	[6.34]		
2	1.40×10^{-3}	3.77×10^{-4}	1.78×10^{-3}	27 %
	[42.67]	[6.29]		
4	2.78×10^{-3}	7.85×10^{-4}	3.57×10^{-3}	28 %
	[42.37]	[6.55]		

Table 5.3: $\%\Delta s$ gives the percentage increase in the volatility measure for bond returns during recessions for the ARCH(1) model with $\rho = 0.20$ and $\alpha = 0.20$, for varying coefficients of relative risk aversion γ. $s_{\exp}$ is a measure of bond return volatility during expansions, while s_{rec} is a measure of bond return volatility during recessions. White adjusted-for-heteroscedasticity t-values are given in brackets below each estimate.

Table 5.3: Volatility of Bond Returns in Recession and Expansion Canonical Constant Variance Model

	$s^{\exp}$	$s^{rec}-s^{\exp}$	s^{rec}	$\%\Delta s$
γ	$\widehat{\beta}_1$	$\widehat{\beta}_2$	$\widehat{\beta}_1+\widehat{\beta}_2$	$\widehat{\beta}_2/\widehat{\beta}_1$
1.5	9.35×10^{-4}	8.29×10^{-6}	9.43×10^{-4}	1 %
	[72.67]	[0.40]		
2	1.86×10^{-3}	2.45×10^{-5}	1.88×10^{-3}	1 %
	[72.26]	[0.60]		
4	3.74×10^{-3}	4.95×10^{-5}	3.79×10^{-3}	1 %
	[72.20]	[0.60]		

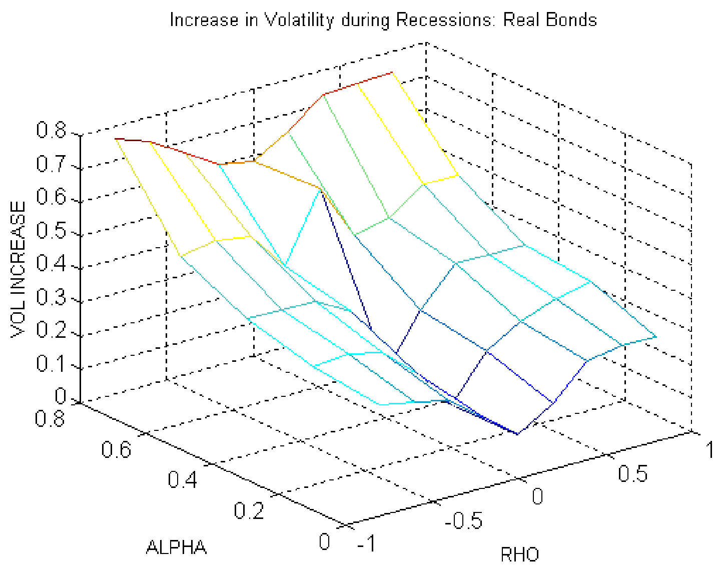
Table 5.4: $\%\Delta s$ gives the percentage increase in volatility of bond returns during recessions for the constant variance model, while γ is the coefficient of relative risk aversion. $s_{\exp}$ is a measure of bond return volatility during expansions, and s_{rec} is a measure of bond return volatility during recessions. White adjusted-for-heteroscedasticity t-values are given in brackets below each estimate.

5.1.4 How important are ρ and α?

A sensitivity analysis was performed in order to investigate the effects of varying the troubling parameters ρ and α widely. Serial correlation ρ was allowed to vary between -0.8 and 0.8, while the ARCH parameter α (serial correlation in volatility) was allowed to vary between 0.0 and 0.80.

Fortunately, the increase in volatility in real bond returns is quite robust to choices of ρ, as can be seen in Figure 1. Thus, unreliable measurement of serial correlation is not so serious, since serial correlation is not an important determinant of volatility asymmetries.

In contrast, the ARCH parameter is indeed an important determinant of volatility asymmetries, confirming the results of the theoretical analysis of the previous section. As can be seen in Figure 3, the increase in volatility during recessions is sharply increasing in α, underlining the crucial role of ARCH in generating volatility asymmetries. Moreover, even low degrees of ARCH (α =0.20) are sufficient to generate significant increases in volatility during recessions. This implies that the amount of ARCH in consumption growth need not be large in order to account for the greater volatility of bond returns during recessions.[12]



Sensitivity Analysis: The percentage increase in volatility during recessions is plotted on the vertical axis over a grid of serial correlations $\rho \in [-0.8, 0.8]$ and ARCH parameters $\alpha \in [0.0, 0.8]$. Clearly, the increase in volatility is sharply increasing in α.

[12] The remaining parameter shown to be important in the theoretical analysis of Section 4 was the mean consumption growth rate μ_x. In the appendix, results of a sensitivity analysis on μ_x are presented. Once again, the theoretical analysis is confirmed: volatility asymmetries are increasing in μ_x. However, volatility asymmetries remain significant at all reasonable growth rates, and only become insignificant for zero growth rates.

5.2 Countercyclical Heteroscedasticity in Nominal Bond Returns

Although the discussion of real bonds is illustrative, the overwhelming majority of bonds traded are nominal bonds. Moreover, Schwert (1989)'s evidence that short-term interest rates are more volatile during recessions refers to nominal short-term rates. Thus it is necessary to extend the analysis to take nominal bonds into account.

A nominal bond paying one unit of currency at date $t+1$ is sold for q_t units of currency at date t. This implies that the real return on the nominal bond is equal to:

$$R^f_{t+1} = \frac{1/p_{t+1}}{q_t/p_t} = \frac{1}{\pi_{t+1} q_t} \tag{31}$$

where p_t is the price level at date t, and $\pi_{t+1} \equiv \frac{p_{t+1}}{p_t}$ is the inflation rate between dates $t+1$ and t. Using the Euler equation (4), this implies that in equilibrium

$$1 = E_t \left\{ M_{t+1} R^f_{t+1} \right\}$$

where M_{t+1} is the pricing kernel. Under log utility this implies

$$q_t = \beta \cdot E_t \left\{ x_{t+1}^{-\gamma} \cdot \frac{1}{\pi_{t+1}} \right\} \tag{32}$$

where x_{t+1} is the stochastic consumption growth rate. If x_{t+1} and π_{t+1} are jointly log-normally distributed, then one may write [following Hansen and Singleton (1983)]

$$\begin{aligned} \log q_t &= \log\beta - \gamma E_t \log x_{t+1} - E_t \log \pi_{t+1} \\ &\quad + \frac{1}{2}\left[\gamma^2 Var_t \log x_{t+1} + Var_t \log \pi_{t+1} + 2\gamma Cov_t\left(\log x_{t+1}, \log \pi_{t+1}\right)\right] \end{aligned}$$

So that the nominal return on a nominal bond $R^{f,n}_{t+1} = -\log q_t$ may be expressed as:

$$\begin{aligned} R^{f,n}_{t+1} &= -\log\beta + \gamma E_t \log x_{t+1} + E_t \log \pi_{t+1} - \frac{\gamma^2}{2} var_t \log x_{t+1} \\ &\quad - \frac{1}{2} var_t \log \pi_{t+1} \underbrace{-\gamma Cov_t\left(\log x_{t+1}, \log \pi_{t+1}\right)}_{\text{inflation risk premium}} \end{aligned} \tag{33}$$

Clearly, Equation (33) shows that inflation and consumption growth play analogous roles in the determination of the nominal bond return $R^{f,n}_{t+1}$. Thus, ARCH in inflation will lead to volatility asymmetries in the same way as ARCH in consumption growth has been shown to do. The only difference lies in the differing role of risk aversion: since the inflation terms $E_t \log \pi_{t+1}$ and $var_t \log \pi_{t+1}$ are not multiplied by any function of γ, the volatility asymmetries due to ARCH in inflation will not be increasing in risk aversion. On the contrary, due to the role of the inflation risk premium, the degree of volatility asymmetry due to inflation actually turns out to be *decreasing* in γ.

To see this, assume that the log inflation and real consumption growth rates follow a bivariate AR(1)-ARCH(1) process with constant correlation between $var_t(\log x_{t+1})$ and $var_t(\log \pi_{t+1})$.[13] This implies that the inflation risk premium may be written as

$$-\gamma\alpha_{x\pi}\sqrt{var_t(\log x_{t+1}) \cdot var_t(\log \pi_{t+1})} \tag{34}$$

For the empirically relevant case of negative ρ_{12}, time-varying inflation risk premia are especially large (and positive) whenever volatility in consumption growth or inflation or both are large. This will tend to counteract the effects of the when it rains it pours mechanism. Whether this inflation premium effect is able to have a significant impact on volatility asymmetries will depend upon the level of risk aversion γ and the correlation ρ_{12}. At low levels of risk aversion, and for the low estimates of ρ_{12} reported below, however, this countervailing effect will turn out to be of relatively minor importance.

5.2.1 Bivariate AR(1)-ARCH(1) Estimates

To calibrate the model, log inflation and consumption growth rates are estimated as a joint AR(1) system:

$$\begin{bmatrix} \log x_{t+1} \\ \log \pi_{t+1} \end{bmatrix} = \begin{bmatrix} c_x \\ c_\pi \end{bmatrix} + \begin{bmatrix} \rho_{xx} & \rho_{x\pi} \\ \rho_{x\pi} & \rho_{\pi\pi} \end{bmatrix} \begin{bmatrix} \log x_t \\ \log \pi_t \end{bmatrix} + \begin{bmatrix} u_{x,t+1} \\ u_{\pi,t+1} \end{bmatrix} \tag{35}$$

where the disturbances are governed by a bivariate ARCH(1) process with constant correlations as:

$$\begin{bmatrix} u_{x,t} \\ u_{\pi,t} \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma^2_{x,t} & \sigma^2_{x\pi,t} \\ \sigma^2_{x\pi,t} & \sigma^2_{\pi,t} \end{bmatrix} \right)$$

and

$$\sigma_{x,t} = \sqrt{\xi_x + \alpha_x u^2_{x,t-1}} \cdot v_{x,t} \qquad v_{x,t} \sim N(0,1) \tag{36}$$

$$\sigma_{\pi,t} = \sqrt{\xi_\pi + \alpha_\pi u^2_{\pi,t-1}} \cdot v_{\pi,t} \qquad v_{\pi,t} \sim N(0,1) \tag{37}$$

$$\sigma_{x\pi,t} = \alpha_{x\pi}\sigma_{\pi,t}\sigma_{x,t} \tag{38}$$

Maximum likelihood estimation of this system yields the following estimates for the serial correlations:

[13] The choice of this formulation, first introduced by Bollerslev (1990), is motivated not only by its tractability. Also attractive is the fact that the constant correlations approach restricts inflation risk premia to be positive for the empirically relevant case where $\rho_{12} < 0$. Note that the more general VECH model of Bollerslev, Engle and Wooldridge (1988) would define the inflation risk premium as $-\gamma[\omega + \alpha u_{xt}u_{\pi t}]$, so that the inflation risk premium might become negative when innovations to the consumption and inflation processes take the same sign.

Bivariate AR(1)-ARCH(1) Estimates

Parameter	Estimate	Parameter	Estimate
ρ_{xx}	-0.2728	α_x	0.0039
$\rho_{\pi\pi}$	0.6629	α_π	0.3075
$\rho_{x\pi}$	-0.0623	$\alpha_{x\pi}$	-0.1960

Note that significant degrees of ARCH volatility and positive serial correlation are found in the log inflation series. In contrast, monthly consumption growth continues to exhibit (probably spurious) negative serial correlation and no statistically significant degree of ARCH. Thus, in nominal bonds, the volatility asymmetries may be fully attributed to ARCH in inflation. These bivariate results are consistent with the results of univariate ARCH LM tests carried out on the same data. ARCH LM tests find evidence of significant ARCH effects in inflation, but not in consumption growth. Remaining parameters for the AR(1)-ARCH(1) process were chosen by exploiting properties of autoregressive processes, to ensure that the constants $(c_x, c_\pi, \xi_x, \xi_\pi)$ are consistent with both the estimated serial correlations and the unconditional moments $(\mu_x, \mu_\pi, \sigma_x, \sigma_\pi)$ reported in the table below.

Unconditional Means of Log Inflation and Consumption Growth

$\log x_t$		$\log \pi_t$	
μ_x	σ_x	μ_π	σ_π
0.0011	0.0040	0.0041	0.0030

5.2.2 Simulation Results: Nominal Bonds

The ARCH(1) model can indeed account for significantly greater nominal bond return volatility during recessions, as can be seen from Table 5.4 below. The increase in volatility during recessions ranged from 32% to 47%, and is somewhat greater than that reported by Schwert (1989) for the long data, as can be seen from the last column of Table 5.4 above. Moreover, this increase is highly statistically significant, as can be seen from the t-values on the estimates of $\widehat{\beta}_2$. The simulation results of Table 5.4 also confirm the theoretical prediction that the increase in volatility would be decreasing in risk aversion γ. This is due to the inflation risk premium, which is not only increasing in γ, but also serves to dampen the volatility asymmetries.

Table 5.4: Nominal Bond Volatility in Recession and Expansion ARCH(1) Model

	$s^{\exp}$	$s^{rec}-s^{\exp}$	s^{rec}	$\%\Delta s$
γ	$\widehat{\beta}_1$	$\widehat{\beta}_2$	$\widehat{\beta}_1+\widehat{\beta}_2$	$\widehat{\beta}_2/\widehat{\beta}_1$
1.5	2.11×10^{-3}	9.87×10^{-4}	3.10×10^{-3}	47 %
	[62.24]	[17.40]		
2.0	3.86×10^{-3}	1.61×10^{-3}	5.57×10^{-3}	42 %
	[76.57]	[19.12]		
4.0	1.16×10^{-2}	3.73×10^{-3}	1.53×10^{-3}	32 %
	[85.50]	[16.50]		
data				
1859-1987				29 %
1953-1987				134 %

Table 5.4: $\%\Delta s$ gives the percentage increase in volatility of nominal bond returns during recessions in the ARCH(1) model with varying risk aversion coefficients γ. A measure of the volatility of returns during expansions is provided by $s^{\exp}$, and s^{rec} measures the volatility of returns during recessions. White adjusted-for-heteroscedasticity t-values are given in brackets below each estimate.

5.3 Countercyclical Heteroscedasticity in Equity Returns

Returning to the real model, the results are similarly positive for equity returns. Time-varying second moments also lead to significantly greater degrees of volatility during recessions in equity returns. Again, the *when it rains it pours* mechanism greatly increases the amount of CCH which the model can account for. To see this, compare the degree of CCH in the constant variance model (middle column of Table 5.5) to the degree of CCH in the ARCH(1) model, shown in the far right column of Table 5.5. [For details on the simulation method, see Appendix C.]

Table 5.5: Countercyclical Heteroscedasticity in Equity Returns ARCH(1) versus no ARCH

	$\%\Delta s$	$\%\Delta s$
γ	no ARCH	ARCH(1)
1.5	68 %	115 %
2	59 %	94 %
4	45 %	37 %
data	1859-1987	61 %
	1920-1952	234 %
	1953-1987	68 %

Table 5.6: Equity Return Volatility in Recession and Expansion ARCH(1) model

	$s^{\exp}$	$s^{rec}-s^{\exp}$	s^{rec}	$\%\Delta s$
γ	$\widehat{\beta}_1$	$\widehat{\beta}_2$	$\widehat{\beta}_1+\widehat{\beta}_2$	$\widehat{\beta}_2/\widehat{\beta}_1$
1.5	0.00200	0.00229	0.00429	115 %
	[39.90]	[23.29]		
2	0.00198	0.00187	0.00386	94 %
	[41.86]	[20.19]		
4	0.00245	0.00088	0.00333	37 %
	[41.85]	[7.62]		

Table 5.6: $\%\Delta s$ gives the percentage increase in volatility of equity returns during recessions in the ARCH(1) model with varying risk aversion coefficients γ. An estimate of the volatility of equity returns during expansions is provided by $s^{\exp}$, and s^{rec} is an estimate of the volatility of equity returns during recessions. White adjusted-for-heteroscedasticity t-values are given in brackets below each estimate.

At low levels of risk aversion, the estimated increase in volatility during recession $\widehat{\beta}_2$ is positive and highly significant for both models.[14] However, the degree of CCH generated by the ARCH(1) model is more than twice as great as that generated by the constant variance model: 140 % in the ARCH(1) case, as opposed to 68 % in the constant variance case for $\gamma = 1.5$. These values compare favorably with those estimated by Schwert (1989) for U.S. data and reported in Table 5.11 below.

Table 5.7: Equity Return Volatility in Recession and Expansion Constant Variance Model

	$s^{\exp}$	$s^{rec}-s^{\exp}$	s^{rec}	$\%\Delta s$
γ	$\widehat{\beta}_1$	$\widehat{\beta}_2$	$\widehat{\beta}_1+\widehat{\beta}_2$	$\widehat{\beta}_2/\widehat{\beta}_1$
1.5	0.00282	0.00192	0.00474	68 %
	[38.25]	[26.08]		
2	0.00347	0.00206	0.00553	59 %
	[38.53]	[25.98]		
4	0.00489	0.00219	0.00708	45 %
	[38.83]	[21.42]		

Table 5.7: $\%\Delta s$ gives the percentage increase in volatility of equity returns during recessions in the constant variance model with varying risk aversion coefficients γ.

[14] That some degree of CCH is also found in non-ARCH equity returns is not surprising. Recall that it was the non-linearity of the ARCH bond return which was driving the CCH result there. Since the non-ARCH equity return is also non-linear, it is plausible that it should also display some amount of CCH.

That the constant variance model also exhibits some degree of cyclical variation in equity return volatility is not surprising, when one considers that non-linearities were seen to be driving the CCH in bond returns. Since constant-variance equity returns are already non-linear, they too can be expected to display some degree of cyclical variation in their volatility. That ARCH(1) equity returns display even greater degrees of CCH reflects the added degree of non-linearity contributed by the time-varying second moments.

Table 5.8: Increase in Equity Return Volatility during Recession U.S. monthly data [Source: Schwert (1989)]

period	**1859-1987**	**1859-1919**	**1920-1952**	**1953-1987**
$\%\Delta s$	61%	-6%	234%	68%

5.4 CCH and Equity Premia

The equity return CCH results should, however, be treated with some of caution. To see why, first note that the degree of CCH which both the ARCH(1) and the constant variance model can generate is *decreasing* in the degree of risk aversion. This, in turn, may be related to the relationship between risk aversion and risk premia.

To be more precise, equity returns may be written as the sum of the bond return and the equity risk premium as

$$r_{t+1}^{e} = r_{t+1}^{f} + \underbrace{\left(r_{t+1}^{e} - r_{t+1}^{f}\right)}_{\text{risk premium}} \tag{39}$$

Bond returns r_{t+1}^{f} are decreasing in the variance of the underlying asset (due to precautionary effects). Risk premia, on the other hand, are *increasing* in the volatility of the underlying asset. The more variable the stream of payoffs, the more a risk averse agent will have to be compensated for holding it. The total effect is ambiguous.

For low levels of risk aversion, equity returns turn out to be highly correlated with bond returns ($corr\left(R_t^e, R_{t+1}^f\right) \approx 0.98$), implying that the total effect has equity returns *decreasing* in the volatility of the underlying asset. Recalling the theoretical discussion of Section 4, it is precisely this negative relationship between volatility and returns which allows to generate the large degrees of CCH documented in Table 5.9. The tight correlation between bond and equity returns is also reflected, however, in the extremely small equity premia generated by the power utility model studied here at low levels of risk aversion. Furthermore, when risk aversion in increased to moderate levels ($\gamma = 4.0$), the CCH generated by the power utility model begins to evaporate. Thus, one might be tempted to suspect that the greater volatility of equity returns in recession is intimately linked to the extremely low - and counterfactual - equity premia associated with the basic Lucas(1978)-Mehra/Prescott (1985) model.

It turns out that this suspicion is unfounded. In Ebell (2000a) I check whether it is possible to generate endogenously *both* large equity premia and more volatile equity returns during recessions. In particular, I extend Cochrane and Campbell (1999)'s habit persistence model to include AR(1)-ARCH(1) consumption growth. It turns out that this habit-plus-ARCH model can indeed account for both realistic equity premia of 6.05 % per annum *and* equity returns that are about 118% more volatile during recessions.

6 Generalization

It is clear that the analytical results presented in section 4 rely primarily on the fact that the endogenously generated AR(1)-ARCH(1) bond return can be expressed as a quadratic function of the innovation. This observation motivates a generalization to all endogenous variables which may be expressed (or approximated) as quadratic functions innovations.

Endogenous variables that are quadratic in the innovations react asymmetrically to positive and negative innovations. Thus, to the extent that a given endogenous variable may be expressed (or approximated) as a quadratic function of the innovation, it should also exhibit asymmetric volatility. More precisely, say that some endogenous variable y_{t+1} may be expressed (or approximated) by a quadratic equation in the innovation to its driving process w_t. In particular, consider an endogenous variable y_t, which may be expressed as

$$y_t = a + bw_t + cw_t^2 \tag{40}$$

where

$$\frac{dy_t}{dw_t} = b + 2cw_t \tag{41}$$

Equation (41) gives the generalized version of the ARCH line of Section 4. Its intercept will be determined by b, while its slope will be determined by $2c$. Figure (8) shows the reaction in y_t to an innovation w_t when the endogenous variable reacts positively to the innovation w_t ($b > 0$), but *negatively* to variance $c < 0$. In this case, the shaded region under the $\frac{dy_{t+1}}{dw_t}$ line between w^- and zero is clearly greater than that between w^+ and zero, reflecting the stronger reaction of y_{t+1} to negative innovations. Thus, y_{t+1} behaves like a bond return, and volatility will tend to be greater during recessions, a tendency which will be reinforced by the fact that the (conditional) mean-inducing innovation will be positive for $b < 0$ and $c > 0$, as stated in the proposition below.

Definition

The (un)conditional mean-inducing innovation ($\widetilde{w}_t^u$) $\widetilde{w}_t$ is the smallest magnitude innovation which causes y_t to take on its (conditional) mean value.

Lemma

$\widetilde{w}_t^u > 0$ and $\widetilde{w}_t > 0$ *whenever b and c are of opposite signs.*

Proof

See Appendix D.■

If, on the other hand, the endogenous variable reacts positively both to variance ($c > 0$) and to the level, the volatility asymmetry is likely to be reversed. To see this, note that in Figure (9), the slope of the $\frac{dy_{t+1}}{dw_t}$ line is positive. Thus, the area under this line will be greater for w^+ than for w^-, and volatility will tend to be greater during expansions. The fact that the (conditional) mean-inducing innovation is negative reinforces the tendency toward procyclical heteroscedasticity. This tendency will be reinforced by the fact that the (conditional) mean-inducing innovation will be negative when $b, c > 0.$, by the Lemma below.

Lemma

$\widetilde{w}_t < 0$ and $\widetilde{w}_t^u < 0$ *whenever* b *and* c *are of the same sign.*

Proof

See Appendix D.

7 Conclusions

The main theoretical contribution of this paper has been to show that introducing time-varying second moments into a consumption CAPM framework can induce cyclical variation in asset returns. In particular, positive serial correlation of growth rates and variances in a growing CCAPM economy lead to countercyclical heteroscedasticity in bond returns. The main quantitative contribution has been to show that the degree of countercyclical heteroscedasticity generated by the model is quantitatively significant and empirically relevant. The increase in volatility of nominal bond returns in the simulated model is similar to that found in the data when the model is calibrated to match US post-war data on consumption growth and inflation. Also, the increase in equity return volatility matches well that found in the data.

Furthermore, the *when it rains it pours* mechanism has the potential to explain cyclical variation in the volatilities of more general economic variables. Although this paper concentrates on explaining countercyclical heteroscedasticity in financial markets, the mechanism is in no way specific to financial markets. All sorts of forcing processes may have time-varying second moments, and ARCH driving processes may be integrated into any number of models. Moreover, there are no limits *per se* on which type of variables are generated endogenously. As shown in Section 6, volatility asymmetries may arise in any variable which may be expressed or approximated as a quadratic function of the innovation to the driving process. As an example, Ebell (2000b) shows how ARCH in the productivity shock of a simple RBC model can lead to production growth which is significantly more volatile during recessions, thereby further broadening the set of stylized facts the *when it rains it pours* mechanism can account for.

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Appendix A Mean-Inducing Innovations I: ARCH Bond Returns

The mean-inducing innovation $\widetilde{u}_t$ is that which induces the bond return to be exactly equal to its mean.

Definition

The (un)conditional mean-inducing innovation ($\widetilde{u}_t^u$) $\widetilde{u}_t$ is the smallest magnitude innovation which causes $r_{t+1}^f(u_t)$ to take on its (conditional) mean value.

$$\begin{aligned} r_{t+1}^f(\widetilde{u}_t^u) &= Er_{t+1}^f \\ r_{t+1}^f(\widetilde{u}_t) &= E_{t-1}r_{t+1}^f \end{aligned}$$

Assume throughout that $\gamma, \alpha, \xi > 0$, so that agents are risk averse, and volatility is positively serially correlated and never expected to be negative.

Appendix A.1
No-ARCH case I: Conditional mean-inducing innovation

In the no-ARCH case, the innovation $\widetilde{u}_t$ which induces the conditional mean bond return $E_{t-1}r_{t+1}^f$ satisfies:

$$\begin{aligned} r_{t+1}^f(\widetilde{u}_t) &= -\log\delta + \gamma\left[c(1+\rho) + \rho^2\log x_{t-1} + \widetilde{u}_t\right] - \frac{\gamma^2}{2}\sigma_x^2 \\ &= -\log\delta + \gamma\underbrace{\left[c(1+\rho) + \rho^2\log x_{t-1}\right]}_{E_{t-1}\log x_{t+1}} - \frac{\gamma^2}{2}\sigma_x^2 = E_{t-1}r_{t+1}^f \end{aligned}$$

so that the mean-inducing innovation is clearly

$$\widetilde{u}_t = 0$$

The idea is simple: whatever has happened up to and including date $t-1$, the conditional expectation of next period's bond return is simply that which results when the next period's innovation takes on its (un)conditional mean value of zero.

Appendix A.2
ARCH case II: Conditional mean-inducing innovation

In the ARCH case, the innovation $\widetilde{u}_t$ which induces the conditional mean bond return $E_{t-1}r^f_{t+1}$ satisfies:

$$\begin{aligned} r^f_{t+1}\left(\widetilde{u}_t\right) &= -\log\delta + \gamma\left[c\left(1+\rho\right) + \rho^2\log x_{t-1} + \rho\widetilde{u}_t\right] - \frac{\gamma^2}{2}\left[\xi + \alpha\widetilde{u}_t^2\right] \\ &= -\log\delta + \gamma\underbrace{\left[c\left(1+\rho\right) + \rho^2\log x_{t-1}\right]}_{E_{t-1}\log x_{t+1}} - \frac{\gamma^2}{2}\underbrace{\left[\xi + \left(\rho^2+\alpha\right)\left(\xi + \alpha u^2_{t-1}\right)\right]}_{var_{t-1}\log x_{t+1}} \\ &= E_{t-1}r^f_{t+1} \end{aligned}$$

Rearranging terms, this implies that the mean-inducing innovation satisfies the following quadratic equation:

$$-c\left(u^2_{t-1}\right) - \gamma\rho\widetilde{u}_t + \frac{\gamma^2}{2}\alpha\left(\widetilde{u}_t\right)^2 = 0 \tag{42}$$

where

$$c\left(u^2_{t-1}\right) \equiv \left(\rho^2+\alpha\right)\left(\xi + \alpha u^2_{t-1}\right) > 0 \qquad \text{for all } u^2_{t-1}$$

Making use of the quadratic formula, one may obtain the roots of equation (42) as

$$\widetilde{u}_t = \frac{-\rho \pm \sqrt{\rho^2 + \gamma^2\alpha\cdot c\left(u^2_{t-1}\right)}}{\gamma\cdot c\left(u^2_{t-1}\right)} \tag{43}$$

Which of the two roots is of smaller magnitude will depend upon the sign of ρ. The following two lemmas derive the conditional mean-inducing innovation for positively and negatively serially correlated growth, respectively.

Lemma

For $\rho > 0$, the conditional mean-inducing innovation $\widetilde{u}_t$ is given by

$$\widetilde{u}_t = \frac{-\rho + \sqrt{\rho^2 + \gamma^2\alpha\cdot c\left(u^2_{t-1}\right)}}{\gamma\cdot c\left(u^2_{t-1}\right)} > 0$$

for all values of u^2_{t-1}.

Proof

Consider first the case when consumption growth is positively serially correlated. The two roots may be expressed as

$$\begin{aligned} \widetilde{u}_{t,1} &= \frac{-\rho - \sqrt{\rho^2 + \gamma^2\alpha\cdot c\left(u^2_{t-1}\right)}}{\gamma\cdot c\left(u^2_{t-1}\right)} < 0 \\ \widetilde{u}_{t,2} &= \frac{-\rho + \sqrt{\rho^2 + \gamma^2\alpha\cdot c\left(u^2_{t-1}\right)}}{\gamma\cdot c\left(u^2_{t-1}\right)} > 0 \end{aligned}$$

where negativity and positivity of each root makes use of the observation that $\sqrt{\rho^2 + \gamma^2\alpha \cdot c\left(u_{t-1}^2\right)} > \rho$. This, together with the fact the $c(\log x_{t-1}) > 0$ for all u_{t-1}^2, also makes it easy to see that $|\widetilde{u}_{t,2}| < |\widetilde{u}_{t,1}|$ so that the conditional mean-inducing innovation is:

$$\widetilde{u}_t = \frac{-\rho + \sqrt{\rho^2 + \gamma^2\alpha \cdot c\left(u_{t-1}^2\right)}}{\gamma \cdot c\left(u_{t-1}^2\right)} > 0$$

■

Lemma

For $\rho < 0$, the conditional mean-inducing innovation $\widetilde{u}_t$ is given by

$$\widetilde{u}_t = \frac{-\rho - \sqrt{\rho^2 + \gamma^2\alpha \cdot c\left(u_{t-1}^2\right)}}{\gamma \cdot c\left(u_{t-1}^2\right)} < 0$$

for all values of u_{t-1}^2.

Proof

Now consider the case when consumption growth is negatively serially correlated. The two roots may be expressed as

$$\begin{aligned}\widetilde{u}_{t,1} &= \frac{-\rho - \sqrt{\rho^2 + \gamma^2\alpha \cdot c\left(u_{t-1}^2\right)}}{\gamma \cdot c\left(u_{t-1}^2\right)} < 0 \\ \widetilde{u}_{t,2} &= \frac{-\rho + \sqrt{\rho^2 + \gamma^2\alpha \cdot c\left(u_{t-1}^2\right)}}{\gamma \cdot c\left(u_{t-1}^2\right)} > 0\end{aligned}$$

where negativity and positivity of each root makes use of the observation that $\sqrt{\rho^2 + \gamma^2\alpha \cdot c\left(u_{t-1}^2\right)} > -\rho$. This, together with the fact the $c(\log x_{t-1}) > 0$ for all u_{t-1}^2, also makes it easy to see that $|\widetilde{u}_{t,1}| < |\widetilde{u}_{t,2}|$ so that the conditional mean-inducing innovation is:

$$\widetilde{u}_t = \frac{-\rho - \sqrt{\rho^2 + \gamma^2\alpha \cdot c\left(u_{t-1}^2\right)}}{\gamma \cdot c\left(u_{t-1}^2\right)} < 0$$

■

Appendix B
Sensitivity Analysis on Mean Consumption Growth μ_x

According to the theoretical analysis of Section 4, the greater is the average growth rate of the economy, the more strongly negative a shock must be (on average) in order to throw the economy into recession. Thus, the greater is μ_x, the more likely are recessions to be associated with the precisely the kind of large negative innovations which favor CCH. Table B1 illustrates this point: when the economy is not growing, there is no significant degree of CCH. Furthermore, the degree of CCH is clearly increasing in the unconditional mean growth rate of the economy μ_x, confirming the analysis in Section 4.

Table B1: Bond Return Volatility in Recession and Expansion ARCH(1) Calibration with $\gamma = 1.5$ and Varying μ_x

	$s^{\exp}$	$s^{rec} - s^{\exp}$	s^{rec}	$\%\Delta s$
μ_x	$\widehat{\beta}_1$	$\widehat{\beta}_2$	$\widehat{\beta}_1 + \widehat{\beta}_2$	$\widehat{\beta}_2/\widehat{\beta}_1$
0.00	9.97×10^{-4}	7.29×10^{-6}	1.00×10^{-3}	0.8 %
	[41.90]	[0.22]		
0.0016	9.26×10^{-4}	3.01×10^{-4}	1.22×10^{-3}	32 %
	[42.08]	[6.93]		
0.0032	9.48×10^{-4}	3.67×10^{-4}	1.31×10^{-3}	39 %
	[40.60]	[6.00]		
0.0063	9.71×10^{-4}	5.30×10^{-4}	1.44×10^{-3}	55 %
	[37.95]	[4.85]		
1859-1987				29 %

Table B1: $\%\Delta s$ gives the percentage increase in volatility of bond returns during recessions in the ARCH(1) model with $\gamma = 2.0$ and varying unconditional mean monthly growth rates μ_x.An estimate of the volatility of bond returns during expansions is provided by $s^{\exp}$, and s^{rec} is an estimate of the volatility of bond returns during recessions. White adjusted-for-heteroscedasticity t-values are given in brackets below each estimate.

Appendix C
Numerical Approximation of Equilibrium Equity Returns

In order to obtain the equilibrium sequence of equity returns, we must first find the sequence of price-dividend ratios $\left\{\frac{p_t}{d_t}\right\}_t$ which satisfies the Euler equation (5). This may be achieved by iterating on equation (5) using the parameterized expectations approach (PEA) developed by Marcet and Marshall (1994). This algorithm finds a parameterization of expectations $\Psi\left(x_t, u_t^2; \psi\right) = E_t\left\{x_{t+1}^{1-\gamma}\left(\frac{p_{t+1}}{d_{t+1}} + 1\right)\right\} = \frac{1}{\beta}\frac{p_t}{d_t}$ which is consistent both with the exogenous growth rates and endogenous price-dividend ratios. That is, the algorithm first assumes some functional form (in our case an exponential one) by which values of the state variables $\left(x_t, u_t^2\right)$ are transformed into expectations:

$$\Psi\left(x_t, u_t^2; \psi\right) = \psi_1 \exp\left\{\psi_2 x_t + \psi_3 u_t^2\right\}$$

Now the series of price-dividend ratios generated by these expectations $\Psi\left(x_t, \widehat{\sigma}_{t+1}^2; \psi\right)$ can be calculated as

$$\frac{p_t}{d_t}(\psi) = \beta \cdot \psi_1 \exp\left\{\psi_2 x_t + \psi_3 u_t^2\right\}$$

Next, the consistency of $\left\{\frac{p_t}{d_t}(\psi)\right\}_t$ needs to be checked. This may be done by imposing rational expectations, and then finding the RE price-dividend ratios as

$$\frac{p_t}{d_t}_{RE} = \delta E_t \left\{ x_{t+1}^{1-\gamma} \left(\frac{p_{t+1}}{d_{t+1}}(\psi) + 1 \right) \right\} \tag{44}$$

Loosely speaking, a fixed point in this algorithm is then the series $\left\{\frac{p_t}{d_t}(\psi)\right\}_t$ which implies itself. In particular, $\left\{\frac{p_t}{d_t}\right\}$ is a PEA solution to the Euler equation if non-linear least squared regressions of the equation

$$\frac{1}{\delta}\frac{p_t}{d_t}_{RE} = \zeta_1 \exp\left\{\zeta_2 x_t + \zeta_3 u_t^2\right\}$$

produce estimates $\left(\widehat{\zeta}_1, \widehat{\zeta}_2, \widehat{\zeta}_3\right)$ which are close enough to those values which generated the price-dividend ratios in the first place, namely (ψ_1, ψ_2, ψ_3).

From the sequence of equilibrium price-dividend ratios, it is easy to recover the sequence of equilibrium equity returns as

$$r_{t+1}^e = x_{t+1} \cdot \frac{\frac{p_{t+1}}{d_{t+1}} + 1}{\frac{p_t}{d_t}} \tag{45}$$

Appendix D
Mean-Inducing Innovations II: General Case

Lemma

The unconditional mean-inducing innovation $\widetilde{w}_t^u$ is positive whenever b and c are of opposite signs, and negative whenever b and c have the same sign.

Proof

Recalling that $Ey_t = a + cE\sigma_w^2$, it is clear that

$$a + cE\sigma_w^2 = a + b\widetilde{w}_t^u + c\left(\widetilde{w}_t^u\right)^2$$

so that

$$-cE\sigma_w^2 + b\widetilde{w}_t^u + c\left(\widetilde{w}_t^u\right)^2 = 0 \tag{46}$$

Making use of the quadratic formula, one can easily see that

$$\widetilde{w}_t^u = \frac{b \pm \sqrt{b^2 + 4c^2 E\sigma_w^2}}{2cE\sigma_w^2}$$

so that

$$\widetilde{w}_{t,1}^u = \frac{b + \sqrt{b^2 + 4c^2 E\sigma_w^2}}{2cE\sigma_w^2} \tag{47}$$

$$\widetilde{w}_{t,2}^u = \frac{b - \sqrt{b^2 + 4c^2 E\sigma_w^2}}{2cE\sigma_w^2} \tag{48}$$

Which of these two roots will have the smaller magnitude will depend upon the signs of b and c. This makes for 4 cases to consider:
Case Ia: $b > 0$, $c < 0$: *This was the case of the bond return. First note that* $b > 0$ *implies that* $b < \sqrt{b^2 + 4c^2 E\sigma_w^2}$. *This, together with* $c < 0$, *implies that* $\widetilde{w}_{t,1}^u < 0$ *and* $\widetilde{w}_{t,2}^u > 0$. *Furthermore,* $\left|\widetilde{w}_{t,2}^u\right| < \left|\widetilde{w}_{t,1}^u\right|$, *so that the mean-inducing innovation is given by:*

$$\widetilde{w}_t^u = \frac{b - \sqrt{b^2 + 4c^2 E\sigma_w^2}}{2cE\sigma_w^2} > 0$$

Case Ib: $b < 0$, $c > 0$ *This is the case of a variable which reacts negatively to an innovation's level, but positively to its variance. Now,* $b < 0$ implies that $-b < \sqrt{b^2 + 4c^2 E\sigma_w^2}$. *This, together with* $c > 0$, *implies that* $\widetilde{w}_{t,1}^u > 0$, *while* $\widetilde{w}_{t,2}^u < 0$. *It also follows that* $\left|\widetilde{w}_{t,1}^u\right| < \left|\widetilde{w}_{t,2}^u\right|$, *so that the mean-inducing innovation is given by*

$$\widetilde{w}_t^u = \frac{b + \sqrt{b^2 + 4c^2 E\sigma_w^2}}{2cE\sigma_w^2} > 0$$

Case IIa: $b < 0$, $c < 0$ *This is the case of the bond return when growth is negatively autocorrelated. As in Case Ia,* $b < 0$ *implies that* $-b < \sqrt{b^2 + 4c^2 E\sigma_w^2}$. *This, together with* $c < 0$, *implies that* $\widetilde{w}_{t,1}^u < 0$ *and* $\widetilde{w}_{t,2}^u > 0$. *Furthermore,* $\left|\widetilde{w}_{t,1}^u\right| < \left|\widetilde{w}_{t,2}^u\right|$, *so that the mean-inducing innovation is given by:*

$$\widetilde{w}_t^u = \frac{b + \sqrt{b^2 + 4c^2 E\sigma_w^2}}{2cE\sigma_{w,t}^2} < 0$$

Case IIb: $b > 0$, $c > 0$ *This is the case of a variable which reacts positively both to an innovation's level and its variance. Now, as in Case Ib,* $b < 0$ *implies that* $b < \sqrt{b^2 + 4c^2 E\sigma_w^2}$. *This, together with* $c > 0$, *implies that* $\widetilde{w}_{t,1}^u > 0$, *while* $\widetilde{w}_{t,2}^u < 0$. *It also follows that* $\left|\widetilde{w}_{t,2}^u\right| < \left|\widetilde{w}_{t,1}^u\right|$, *so that the mean-inducing innovation is given by*

$$\widetilde{w}_t^u = \frac{b - \sqrt{b^2 + 4c^2 E\sigma_w^2}}{2cE\sigma_w^2} < 0$$

■

Lemma

The conditional mean-inducing innovation $\widetilde{w}_t$ *is positive whenever b and c are of opposite signs, and negative whenever b and c have the same sign.*

Proof

Recalling that $E_{t-1}y_t = a + cE_{t-1}\sigma_{w,t}^2$, *it is clear that*

$$a + cE_{t-1}\sigma_{w,t}^2 = a + b\widetilde{w}_t + c\left(\widetilde{w}_t\right)^2$$

so that

$$-cE_{t-1}\sigma_{w,t}^2 + b\widetilde{w}_t + c\left(\widetilde{w}_t\right)^2 = 0 \tag{49}$$

The rest of the proof is completely analogous to that of the previous Lemma.■

Figure 1
ARCH and Constant Variance Bond Returns

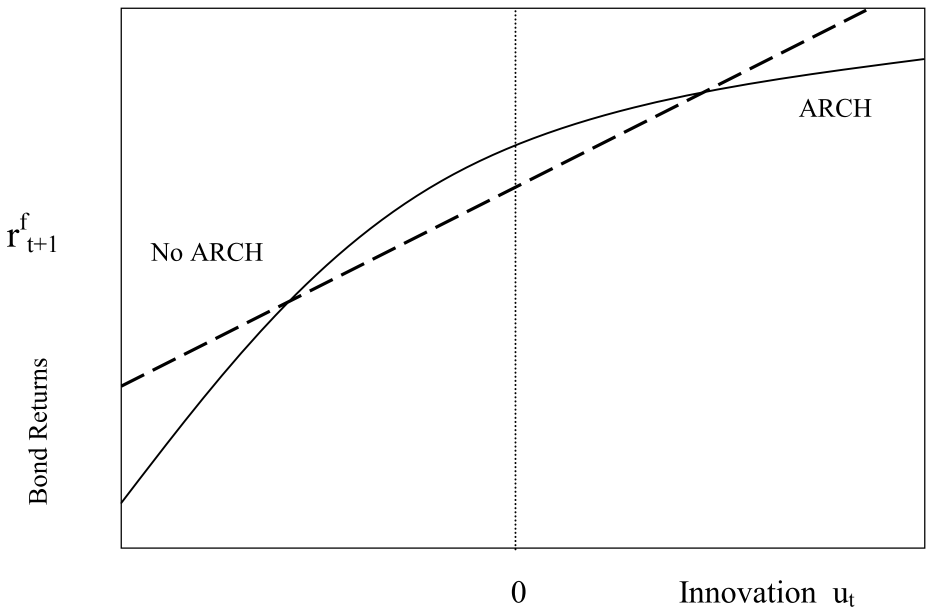

Figure 2
Reaction of Bond Returns to Innovations u(t)

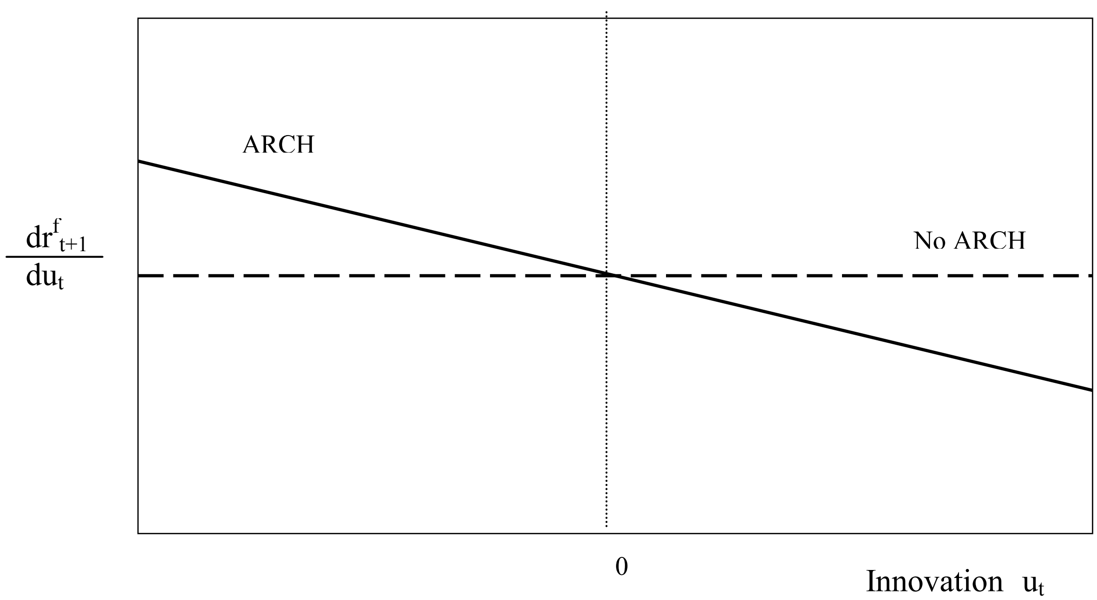

Figure 3
Absolute Deviation of Constant Variance Bond Return due to u(t)

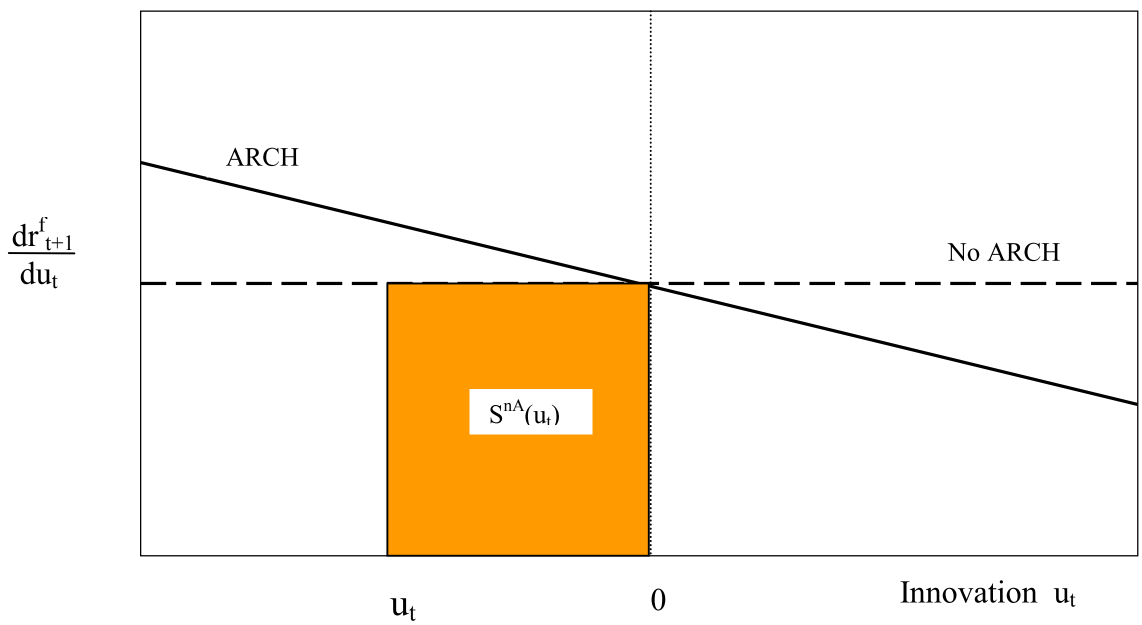

Figure 4
Absolute Deviation of Constant Variance Bond Return due to u+ and u-

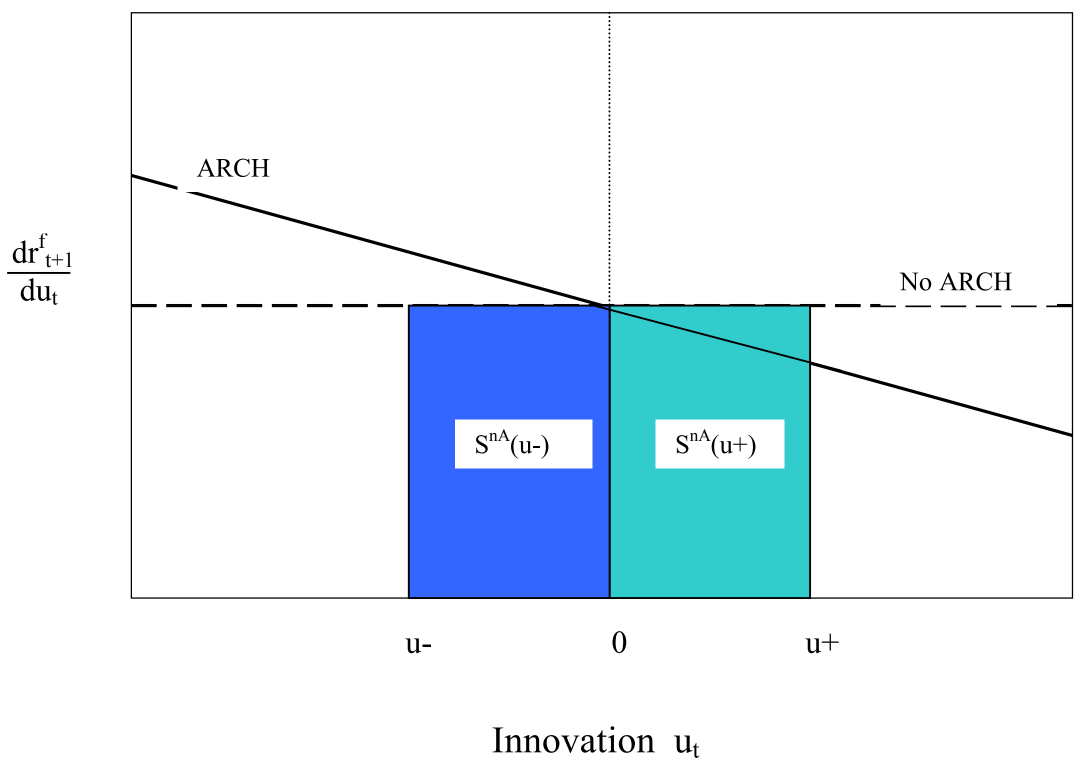

Figure 5
Absolute Deviation of ARCH Bond Return from $r^f(0)$

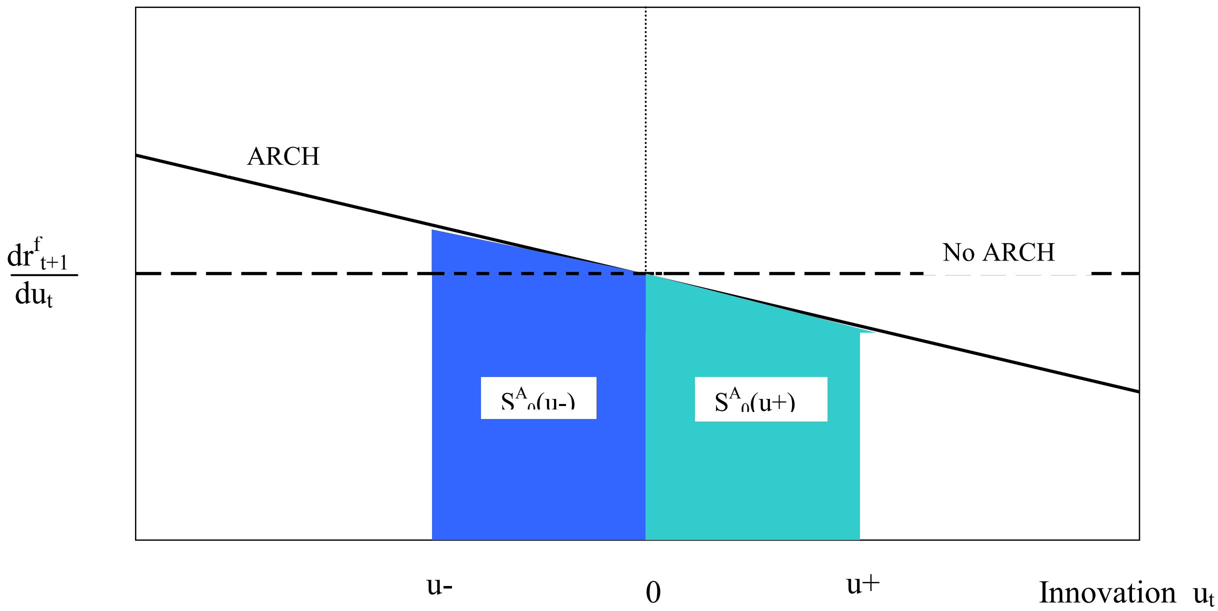

Figure 6
Absolute Deviation of ARCH Bond Return due to u+ and u-

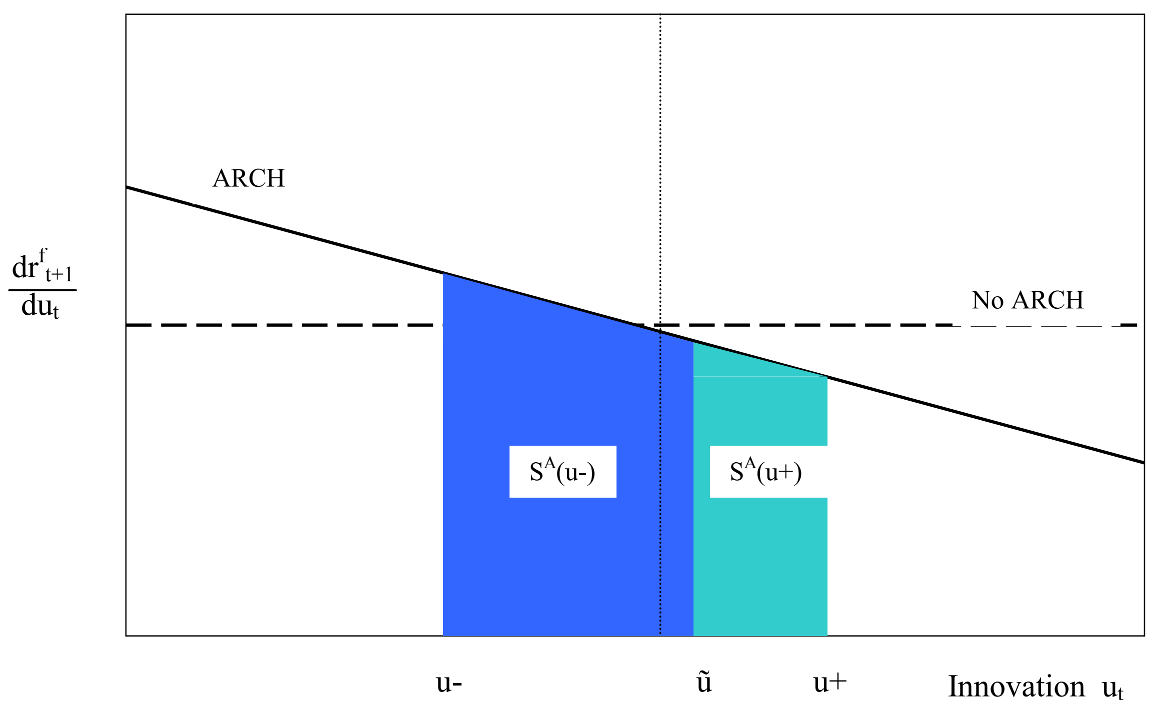

Figure 7
Recessionary and Expansionary Innovations

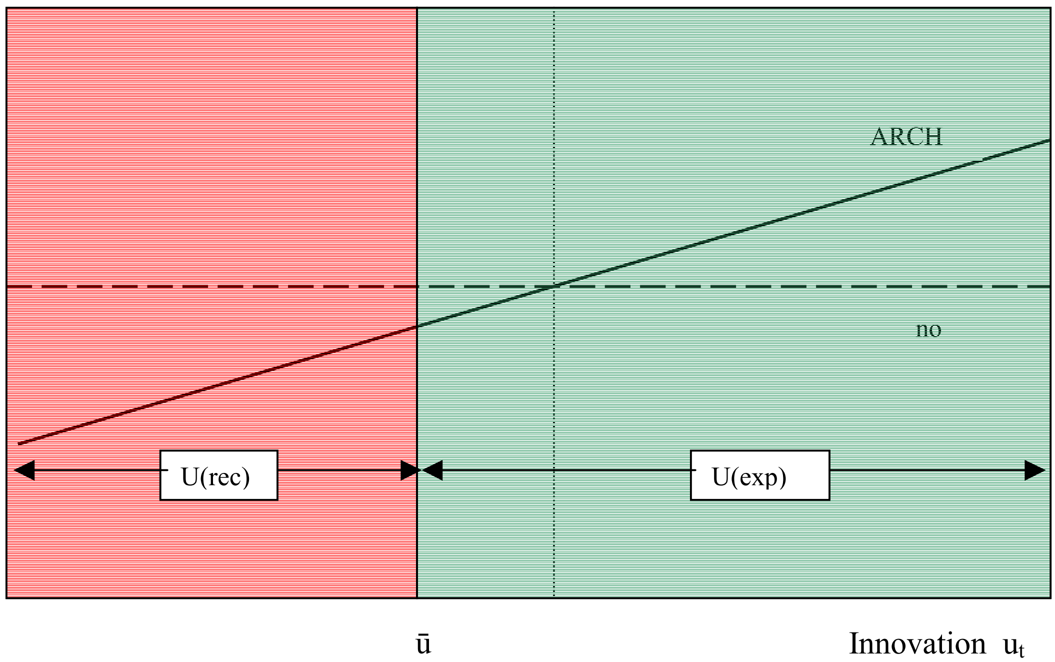

Figure 8
Reaction of y_{t+1} to Innovations when b>0 and c<0

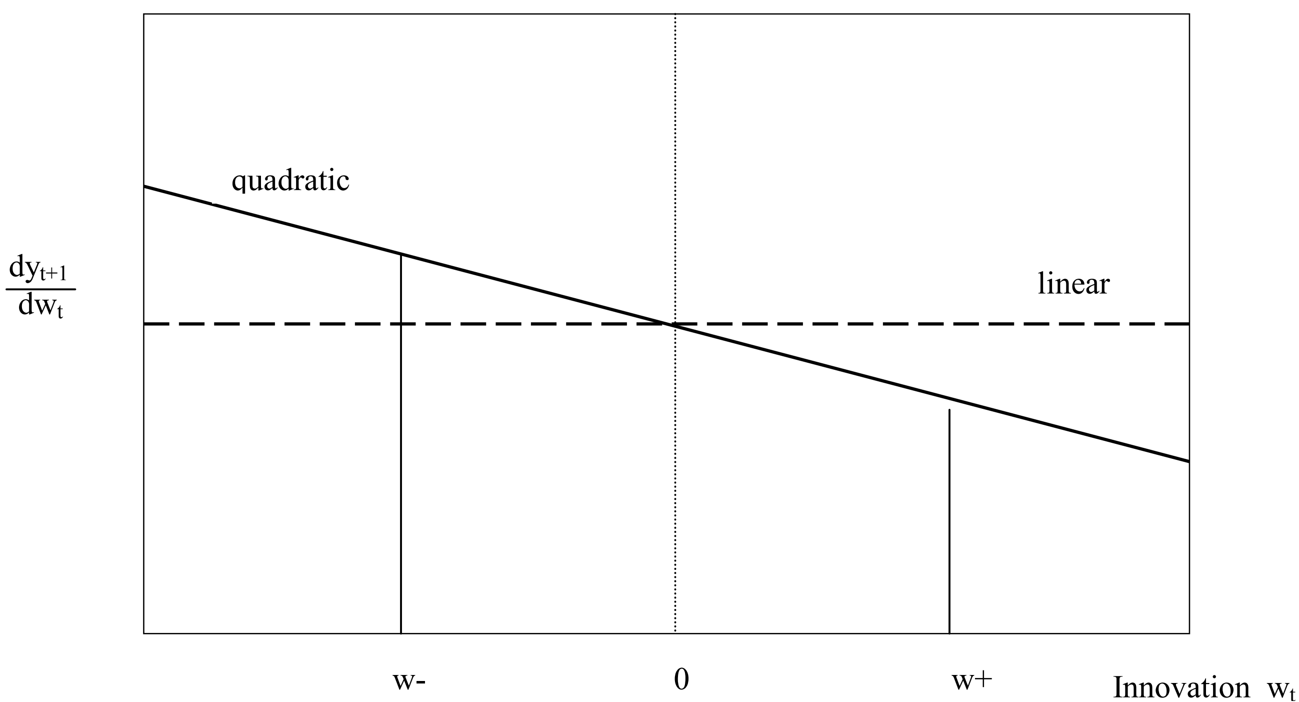

Figure 9
Reaction of y_{t+1} to Innovations when b > 0 and c > 0

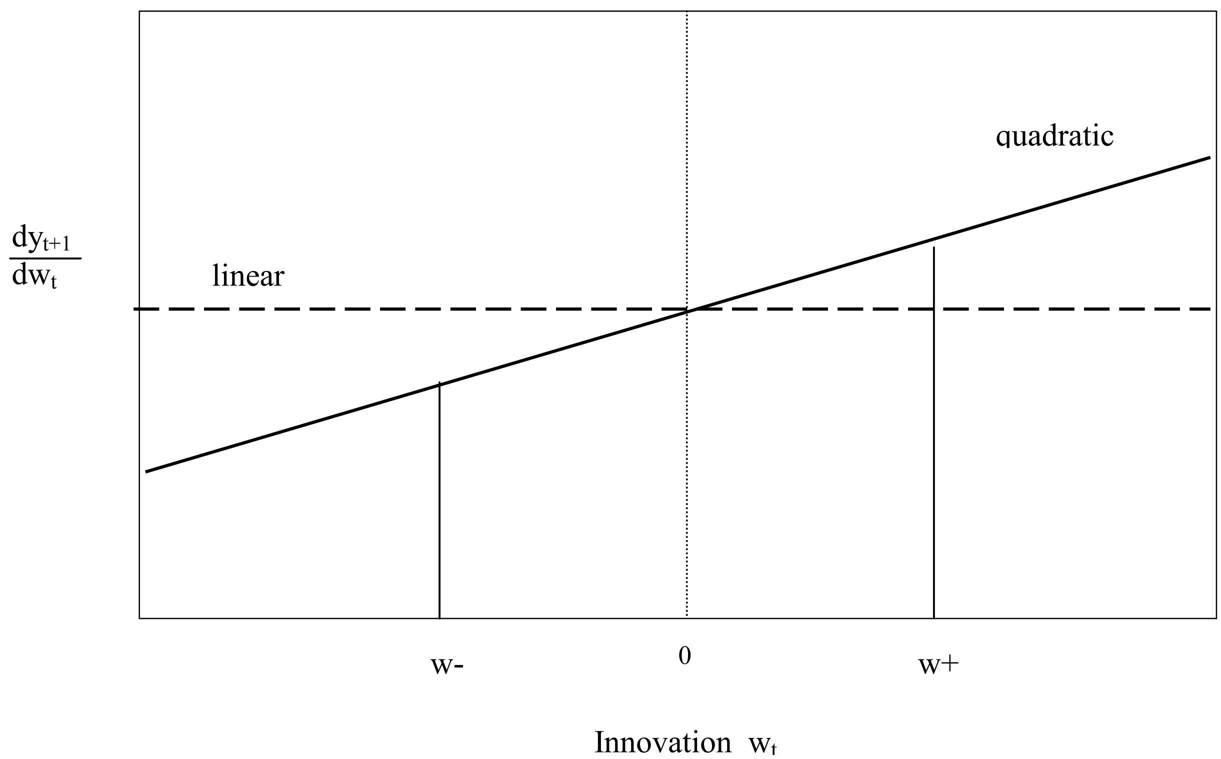

Figure 10
Mean Inducing Innovation