Studienzentrum Gerzensee Doctoral Program in Economics
Midterm Econometrics Exam

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July 28-29, 2000

Instructions:

Write your Identification Number in the space provided below. (Don't give us your name, just your ID number.)

ID Number: SKETCH of ANSWERS

There are 120 points on this 120 minute exam. The number of possible points for each problem is shown in parentheses. If you need additional space use the back of the exam sheet. Feel free to use your notes and any textbooks that you may find useful.

1. (10 Points) Suppose that the 3×1 vector X is distributed $N(0, I_3)$, and let V be a 2×3 matrix that satisfies $VV' = I$. Let $Y = VX$.

   (a) Show that $Y'Y \sim \chi_2^2$

   (i) $VX \sim N(0, VV') \Rightarrow Y \sim N(0, I_2)$

   (ii) $Y'Y \sim \chi^2_2$ follows directly

   (b) Find $Prob(Y'Y > 6)$

   From the tables

   $P(Y'Y > 6) = .05$

2. (20) Consider the linear regression model

$$y_i = x_i\beta + \varepsilon_i$$

For simplicity assume that $\{x_i\}$ is a sequence of constants (i.e., not random) and assume that $\{\varepsilon_i\}$ are i.i.d. with mean 0 and variance σ^2. Consider the two estimators

$$\widehat{\beta}_1 = \left(\sum_{i=1}^{n} x_i^2\right)^{-1} \sum_{i=1}^{n} x_i y_i$$

and

$$\widehat{\beta}_2 = \left(\sum_{i=1}^{n} x_i\right)^{-1} \sum_{i=1}^{n} y_i$$

(where we have assumed that $\sum_{i=1}^{n} x_i^2 \neq 0$ and $\sum_{i=1}^{n} x_i \neq 0$).

(a) (8 points) Find the means and variances of $\widehat{\beta}_1$ and $\widehat{\beta}_2$.

$$\widehat{\beta}_1 = \left(\sum_{i=1}^{n} x_i^2\right)^{-1} \sum_{i=1}^{n} x_i \left(x_i\beta + \varepsilon_i\right) = \beta + \left(\sum_{i=1}^{n} x_i^2\right)^{-1} \sum_{i=1}^{n} x_i\varepsilon_i$$

and

$$\widehat{\beta}_2 = \left(\sum_{i=1}^{n} x_i\right)^{-1} \sum_{i=1}^{n} \left(x_i\beta + \varepsilon_i\right) = \beta + \left(\sum_{i=1}^{n} x_i\right)^{-1} \sum_{i=1}^{n} \varepsilon_i$$

therefore

$$\begin{aligned}
E\left[\widehat{\beta}_1\right] &= \beta + \left(\sum_{i=1}^{n} x_i^2\right)^{-1} \sum_{i=1}^{n} x_i E\left[\varepsilon_i\right] = \beta \\
E\left[\widehat{\beta}_2\right] &= \beta + \left(\sum_{i=1}^{n} x_i\right)^{-1} \sum_{i=1}^{n} E\left[\varepsilon_i\right] = \beta \\
V\left[\widehat{\beta}_1\right] &= V\left[\left(\sum_{i=1}^{n} x_i^2\right)^{-1} \sum_{i=1}^{n} x_i\varepsilon_i\right] = \left(\sum_{i=1}^{n} x_i^2\right)^{-2} V\left[\sum_{i=1}^{n} x_i\varepsilon_i\right] \\
&= \left(\sum_{i=1}^{n} x_i^2\right)^{-2} \sum_{i=1}^{n} x_i^2\sigma^2 = \left(\sum_{i=1}^{n} x_i^2\right)^{-1} \sigma^2 \\
V\left[\widehat{\beta}_2\right] &= V\left[\left(\sum_{i=1}^{n} x_i\right)^{-1} \sum_{i=1}^{n} \varepsilon_i\right] = \left(\sum_{i=1}^{n} x_i\right)^{-2} n\sigma^2
\end{aligned}$$

Also note that $\widehat{\beta}_2$ is the OLS estimator.

(b) (5 points) Consider a third estimator $\widehat{\beta}_3. = a \cdot \widehat{\beta}_1 + (1-a) \cdot \widehat{\beta}_2$. Among the values of a (if any) that result in $\widehat{\beta}_3$ being unbiased, what value of a will result in an estimator with the lowest variance?

$a = 1$ by the Gauss Markow Theorem

(c) (7 points) Now imagine that x_i is $i.i.d$ and independent of the sequence of ε's. Find the mean and the variance of $\widehat{\beta}_2$? (Assume that $\widehat{\beta}_2$ is well–defined with probability 1).

For $j = 1, 2$ we have

$$E\left[\widehat{\beta}_j\right] = E\left[E\left[\widehat{\beta}_j \middle| x_1, x_2, ..., x_n\right]\right] = E\left[\beta\right] = \beta$$

and

$$\begin{aligned} V\left[\widehat{\beta}_j\right] &= E\left[V\left[\widehat{\beta}_j \middle| x_1, x_2, ..., x_n\right]\right] + V\left[E\left[\widehat{\beta}_j \middle| x_1, x_2, ..., x_n\right]\right] \\ &= E\left[V\left[\widehat{\beta}_j \middle| x_1, x_2, ..., x_n\right]\right] + V\left[\beta\right] \\ &= E\left[V\left[\widehat{\beta}_j \middle| x_1, x_2, ..., x_n\right]\right] \end{aligned}$$

so

$$\begin{aligned} V\left[\widehat{\beta}_1\right] &= E\left[\left(\sum_{i=1}^{n} x_i^2\right)^{-1}\right]\sigma^2 \\ V\left[\widehat{\beta}_2\right] &= E\left[\left(\sum_{i=1}^{n} x_i\right)^{-2}\right]n\sigma^2 \end{aligned}$$

3. (20) Suppose that Y_t follows a $MA(2)$ model:

$$Y_t = \varepsilon_t - .2\varepsilon_{t-1} + .4\varepsilon_{t-2}$$

where $\varepsilon_t \sim iidN(0,1)$. You need to make a forecast of Y_{T+1}, but the only piece of information that you have is $Y_T = 2.0$.

(a) (10) What is your forecast value of Y_{T+1}?

$E(Y_{T+1} \mid Y_T) \rightsquigarrow$ optimal (min MSE) Forecast

$E(Y_{T+1} \mid Y_T) = \beta Y_T$ where

$\beta = \frac{COV(Y_{T+1}, Y_T)}{VAR(Y_T)}$ (from joint Normality)

$COV(Y_{T+1}, Y_T) = -.2 - .08 = -.28$

$VAR(Y_T) = 1 + .04 + .16 = 1.20$ $\Rightarrow$

$\beta = -\frac{.28}{1.20}$

(b) (5) What is the variance of the forecast error associated with this forecast?

$VAR(Y_{T+1} / Y_T) =$

$VAR(Y_{T+1}) - \frac{(COV(Y_{T+1}, Y_T))^2}{VAR(Y_T)} =$

$1.20 - \frac{(.28)^2}{1.20} = 1.13$

(c) (5) Suppose that you used data $\{Y_t\}_{t=1}^{T}$ to construct your forecast of Y_{T+1}, and suppose that T was large. What would be the variance of the forecast error? (Does your answer depend on the invertibility of the MA process?)

Assuming that the process was invertible

$$E(Y_{T+1} \mid \{Y_t\}_{t=-\infty}^{T}) = -.2\,\varepsilon_T + .4\,\varepsilon_{T-1}$$

could be determined from lagged Y's

Thus variance of forecast error is 1.

(Forecast error is ε_{T+1})

(You can verify that this process is invertible!)

4. (20) An economist runs two regressions

$$y_i = \beta_0 + x_{1i}\beta_1 + x_{2i}\beta_2 + x_{3i}\beta_3 + \varepsilon_i \tag{1}$$

and

$$y_i = \beta_0 + x_{1i}\beta_1 + \nu_i \tag{2}$$

The results are summarized in the following table:

Equation 1.		
Variable	Coefficient	Std. Error
β_0	1.20	0.18
β_1	0.61	0.09
β_2	-0.08	0.18
β_3	0.42	0.17
R-squared	0.204	
Sum squared resid	106.93	
Number of observations	200	
Equation 2.		
Variable	Coefficient	Std. Error
β_0	1.43	0.11
β_1	0.61	0.09
R-squared	0.178	
Sum squared resid	110.41	
Number of observations	200	

Answer the following questions under the assumption that the usual regularity conditions for OLS are satisfied.

(a) (3 points) Consider Equation 1. Construct a 95% confidence interval for β_1

$$0.61 \pm 1.96 \cdot 0.09 \approx (0.43, 0.79)$$

1. **(b)** (3 points) Test whether β_2 in Equation 1 equals 0 (against the alternative that it differs from 0). Test at a 5% level of significance.

$$T = \frac{-0.08 - 0}{0.18} \approx -0.44 \quad \text{do not reject}$$

1. **(c)** (3 points) Test whether β_3 in Equation 1 equals 0 (test at a 5% level of significance)..

$$T = \frac{0.42 - 0}{0.17} \approx 2.47 \quad \text{reject}$$

(d) (6 points) Test the hypothesis that both β_2 and β_3 in Equation 1 equal 0 (against the alternative that it differs from 0). Test at a 5% level of significance

$$F = \frac{(110.41 - 106.93)/2}{106.93/196} = 3.1894 \quad \text{reject}$$

1. (e) (5 points) Consider Equation 1. Test the hypothesis that $V[\varepsilon_i] \geq 1$ (test at a 5% level of significance).

Recall that

$$s^2 \sim \frac{\sigma^2 \chi^2 (n-k)}{n-k}$$

where $\sigma^2 = V[\varepsilon]$. So if $\sigma^2 = 1$

$$s^2 \sim \frac{\chi^2 (n-k)}{n-k}$$

and we would reject σ^2 if $(n-k)\, s^2$ is small compared to a $\chi^2 (n-k)$ random variable. Here $n - k = 196$, and since many books do not have a table for a $\chi^2 (196)$ it is useful to use the following asymptotically valid approximation. A $\chi^2 (n-k)$ random variable can be written as

$$\sum_{i=1}^{n-k} Z_i^2$$

where Z_i is a sequence of i.i.d. $N(0,1)$ random variables. From the central limit theorem, we have

$$\frac{1}{\sqrt{n-k}} \sum_{i=1}^{n-k} \left(Z_i^2 - 1\right)/2 \xrightarrow{\sim} N(0,1)$$

and hence

$$\frac{1}{\sqrt{n-k}} \left((n-k)\, s^2 - (n-k)\right)/2 \xrightarrow{\sim} N(0,1)$$

Since $(n-k)\, s^2$ is 106.93 (Sum squared resid), the realized value of the left hand side variable is -4.3 and we clearly reject.

5. (30) Consider two random variables X and Y. The distribution of X is given by the uniform distribution $X \sim U[0.5, 1.0]$ and the distribution of Y conditional on X is given by the normal $Y|X \sim N(0, \lambda X^2)$.

(a) (5) Provide an expression for the joint density of X and Y, say $f_{X,Y}(x, y)$.

$f_{X,Y}(x,y) = f_{Y/X}(y|x) \cdot f_X(x)$

$f_{Y/X}(y|x) \rightarrow \frac{1}{\sqrt{2\pi}} \left(\frac{1}{\lambda X^2}\right)^{\frac{1}{2}} e^{-\frac{1}{2}\frac{1}{\lambda X^2} y^2}$

$f_X(x) \rightarrow \begin{cases} 2 & 0.5 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

(b) (5) Prove that $Z = \frac{Y}{X}$ is distributed as $Z \sim N(0, \lambda^2)$.

Conditional on X $\quad Z/X \sim N(0, \lambda^2)$

Since $N(0, \lambda^2)$ does not depend on X,

this is the unconditional distribution as well.

(Note typo --- $Z \sim N(0, \lambda)$ (Not λ^2))

(c) (5) You have a sample $\{X_i, Y_i\}_{i=1}^N$. Derive the log likelihood function $L(\lambda| \{X_i, Y_i\}_{i=1}^N)$.

From (a) Log Likelihood

$$-\frac{N}{2}\log(2\pi) - \frac{1}{2}\sum \log X_i^3 - \frac{N}{2}\log(\lambda) - \frac{1}{2}\frac{1}{\lambda}\sum_{i=1}^{n}\left(\frac{Y_i}{X_i}\right)^2$$

For $\frac{1}{2} \leq X_i \leq 1$ $i = 1, \cdots, N$, $-\infty$ otherwise

$$\hat{\lambda}_{MLE} = \frac{1}{N}\sum\left(\frac{Y_i}{X_i}\right)^2$$ by the usual calculation

(d) (8) Construct the MLE of λ.

(e) (7) Suppose that $N = 10$ and $\sum_{i=1}^{10}(\frac{Y_i}{X_i})^2 = 4.0$. Construct a 95% confidence interval for λ.

Let $g = \sum(\frac{Y_i}{X_i})^2$. Note

$$\frac{g}{\lambda} \sim \chi^2_{10}$$

Thus

$$P\left[3.25 \leq \frac{g}{\lambda} \leq 20.48\right] = .95$$

From χ^2 TABLE

Thus $$P\left[\frac{g}{20.48} \leq \lambda \leq \frac{g}{3.25}\right] = .95$$

$$P\left[.195 \leq \lambda \leq 1.23\right] = .95$$

$(g = 4.0)$

2. (20 points) Suppose that you have independent and identically distributed observations of (y_t, x_t', z_t) from

$$y_t = x_t'\beta + (1 + z_t)\varepsilon_t, \qquad t = 1, 2, ..., T$$

where ε_t has mean mean 0 and variance σ^2 and is independent of (x_t', z_t). Let $\widehat{\beta}$ be the OLS estimator in a regression of y_t on x_t, and assume that all relevant moments exist.

(a) (5 points) Is $\widehat{\beta}$ consistent? Why?

Yes. Heteroskedasticity does not ruin the consistency of OLS

(b) (10 points) Find the asymptotic distribution of $\widehat{\beta}$.

We know from class that

$$\begin{aligned}
\sqrt{n}\left(\widehat{\beta}_n - \beta\right) &= \left[\frac{1}{n}\sum_{i=1}^{n} x_i x_i'\right]^{-1}\left[\frac{1}{\sqrt{n}}\sum_{i=1}^{n} x_i\left(y - x_i'\beta\right)\right] \\
&\xrightarrow{\sim} E\left[x_i x_i'\right]^{-1} \times N\left(0, E\left[x_i\left(y - x_i'\beta\right)^2 x_i'\right]\right)
\end{aligned}$$

Here

$$\begin{aligned}
E\left[x_i\left(y - x_i'\beta\right)^2 x_i'\right] &= E\left[x_i\left(1 - z_i\right)^2 \varepsilon_i^2 x_i'\right] \\
&= E\left[E\left[x_i\left(1 - z_i\right)^2 \varepsilon_i^2 x_i' \middle| x_i, z_i\right]\right] \\
&= \sigma^2 E\left[\left(1 - z_i\right)^2 x_i x_i'\right]
\end{aligned}$$

so

$$\sqrt{n}\left(\widehat{\beta}_n - \beta\right) \xrightarrow{\sim} N\left(0, \sigma^2 E\left[x_i x_i'\right]^{-1} E\left[\left(1 - z_i\right)^2 x_i x_i'\right] E\left[x_i x_i'\right]^{-1}\right)$$

(c) (5 points) How would you estimate σ^2 consistently? (Provide an outline of a prrof of consistency of your proposed estimator)

If β were known, then one could construct

$$\frac{\frac{1}{n}\sum_{i=1}^{n}(y_i - x_i'\beta)^2}{\frac{1}{n}\sum_{i=1}^{n}(1+z_i)^2} \xrightarrow{p} \frac{E\left[(y_i - x_i'\beta)^2\right]}{E\left[(1+z_i)^2\right]} = \frac{\sigma^2 E\left[(1+z_i)^2\right]}{E\left[(1+z_i)^2\right]} = \sigma^2$$

(provided that z_i is not equal to -1 with probability 1). Of course, we do not know β, so we would construct

$$\frac{\frac{1}{n}\sum_{i=1}^{n}\left(y_i - x_i'\widehat{\beta}\right)^2}{\frac{1}{n}\sum_{i=1}^{n}(1+z_i)^2}$$

The remaining step in a consistency proof is then to show that under suitable regularity conditions,

$$\frac{\frac{1}{n}\sum_{i=1}^{n}(y_i - x_i'\beta)^2}{\frac{1}{n}\sum_{i=1}^{n}(1+z_i)^2} - \frac{\frac{1}{n}\sum_{i=1}^{n}\left(y_i - x_i'\widehat{\beta}\right)^2}{\frac{1}{n}\sum_{i=1}^{n}(1+z_i)^2} \xrightarrow{p} 0$$

Let $v_i = y_i - x_i'\beta$. The left hand side then is

$$\begin{aligned}
& \frac{1}{\frac{1}{n}\sum_{i=1}^{n}(1+z_i)^2}\left(\frac{1}{n}\sum_{i=1}^{n} v_i^2 - \frac{1}{n}\sum_{i=1}^{n}\left(y_i - x_i'\widehat{\beta}\right)^2\right) \\
= & \frac{1}{\frac{1}{n}\sum_{i=1}^{n}(1+z_i)^2}\left(\frac{1}{n}\sum_{i=1}^{n} v_i^2 - \frac{1}{n}\sum_{i=1}^{n}\left(v_i + x_i'\left(\widehat{\beta}-\beta\right)\right)^2\right) \\
= & \frac{1}{\frac{1}{n}\sum_{i=1}^{n}(1+z_i)^2}\left(\left(-\frac{2}{n}\sum_{i=1}^{n} v_i x_i'\right)\left(\widehat{\beta}-\beta\right) - \left(\widehat{\beta}-\beta\right)'\left(\frac{1}{n}\sum_{i=1}^{n} x_i x_i'\right)\left(\widehat{\beta}-\beta\right)\right)
\end{aligned}$$

Since $\widehat{\beta} - \beta \xrightarrow{p} 0$ and all the averages converge to their expectations, the result follows.